DESIGNING AN EXTENDED KALMAN FILTER FOR A STELLAR AIDED STRAPDOWN INERTIAL ATTITUDE REFERENCE

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January 16, 2016

ABSTRACT

Stellar aiding of inertial navigation systems (INSs) is one of the early methods for compensating attitude drift in precision INS applications. Originally applied in gimbaled INS applications, a star-tracker and gyros were mounted on a "stable element", a physical platform surrounded by mechanical gimbals, the platform controlled by gimbal-axis torque motors to a prescribed angular orientation measured by the gyros. Star-tracker readings to known stars provided an accurate measure of stable platform attitude error, the error signal used to correct platform attitude. For airborne applications, stellar aiding has long since been replaced by less costly GPS (Global Positioning System) Kalman filter aiding, however, there still remain a few select applications where stellar attitude updating is still in use (e.g., ballistic missile guidance following launch, deep-space satellite attitude referencing). Because of its simple yet comprehensive structure, stellar Kalman filter aiding of strapdown inertially computed attitude provides an excellent vehicle for analyzing the structure and formulation of Extended Kalman Filters in general, the subject of this article. Appendix A provides an overview of the general structure of Extended Kalman Filters as a reference for analytical developments to follow.

MATHEMATICAL NOTATION

$\underline{V}$ = Vector without specific coordinate frame designation. A vector is a parameter that has length and direction. Vectors used in the paper are classified as "free vectors", hence, have no preferred location in coordinate frames in which they are analytically described.

$(\underline{V}\times)$ = Cross-product matrix operator form of $\underline{V}$ defined such that for any vector $\underline{W}$, $(\underline{V}\times)\,\underline{W} = \underline{V}\times\underline{W}$.

$(\underline{V}\,.)$ = Dot-product matrix operator form of $\underline{V}$ defined such that for any vector $\underline{W}$, $(\underline{V}\,.)\,\underline{W} = \underline{V}\,.\,\underline{W}$.

$\underline{V}^{A}$ = Column matrix with elements equal to the projection of $\underline{V}$ on coordinate frame A axes. The projection of $\underline{V}$ on each frame A axis equals the dot product of $\underline{V}$ with a unit vector parallel to that coordinate axis.

$\left(\underline{V}^{A}\times\right)$ = Skew symmetric (or cross-product) form of $\underline{V}^{A}$ represented by the square matrix $\begin{bmatrix} 0 & -V_{ZA} & V_{YA} \\ V_{ZA} & 0 & -V_{XA} \\ -V_{YA} & V_{XA} & 0 \end{bmatrix}$ in which V_{YA}, V_{YA}, V_{ZA} are the components of $\underline{V}^{A}$. The matrix product of $\left(\underline{V}^{A}\times\right)$ with another A frame vector equals the cross-product of $\underline{V}^{A}$ with the vector in the A frame, i.e.: $\left(\underline{V}^{A}\times\right)\underline{W}^{A} = \underline{V}^{A}\times\underline{W}^{A}$.

$\left(\underline{V}^{A}\cdot\right)$ = Transpose (or dot-product) form of $\underline{V}^{A}$ represented by the row matrix $\left[V_{XA} \quad V_{YA} \quad V_{ZA}\right]$ in which V_{YA}, V_{YA}, V_{ZA} are the components of $\underline{V}^{A}$. The matrix product of $\left(\underline{V}^{A}\cdot\right)$ with another A frame vector equals the dot-product of $\underline{V}^{A}$ with the vector in the A frame, i.e.: $\left(\underline{V}^{A}\cdot\right)\underline{W}^{A} = \underline{V}^{A}\cdot\underline{W}^{A}$.

$C_{A_2}^{A_1}$ = Direction cosine matrix that transforms a vector from its coordinate frame A_2 projection form to its coordinate frame A_1 projection form, i.e.: $\underline{V}^{A1} = C_{A2}^{A1}\,\underline{V}^{A2}$. The columns of $C_{A_2}^{A_1}$ are projections on A_1 axes of unit vectors parallel to A_2 axes. The rows of $C_{A_2}^{A_1}$ are projections on A_2 axes of unit vectors parallel to A_1 axes. An important property of $C_{A_2}^{A_1}$ is that its inverse equals its transpose.

$\dot{(\;)} = \frac{d(\;)}{dt}$ = Derivative of ( ) with respect to time t.

$\widehat{(\;)}$ = Measured or computed (from measurements) value of parameter ( ) that, in contrast with the idealized error free value ( ), contains errors.

COORDINATE FRAMES

B = Strapdown inertial sensor coordinates (“body frame”) with axes parallel to nominal right handed orthogonal sensor input axes.

I = Non-rotating inertial coordinate frame.

STRAPDOWN INERTIAL ATTITUDE UPDATING

Attitude in a strapdown inertial system is calculated by integrating the following classic form (or quaternion equivalent) [1, Eq. (1)]:

$$\dot{C}_B^I = C_B^I \left(\underline{\omega}^B \times \right) \quad (1)$$

where

$\underline{\omega}^B$ = Strapdown gyro sensed angular rate vector in B frame coordinates.

The version of (1) that would be implemented in a strapdown computer would be

$$\dot{\hat{C}}_B^I = \hat{C}_B^I \left(\hat{\underline{\omega}}^B \times \right) \quad (2)$$

STAR-TRACKER OBSERVATION VECTOR

A star-tracker provides a means of measuring the angular orientation of a unit vector pointing to an observable star for comparison with the star's known location provided in I frame coordinates by a star catalog within the INS. Comparison of the measured star-tracking vector with the catalog expectation provides a measurement of star-tracker mount angular error perpendicular to the star sighting direction. The process is illustrated analytically next.

Assume that a unit direction vector to a known star is detected with a star-tracker and its output transformed to the B frame. Using the $\hat{C}_B^I$ matrix, the inertial system can calculate the detected star-pointing vector components in the I frame:

$$\hat{\underline{u}}_{Star}^I = \hat{C}_B^I \, \hat{\underline{u}}_{Star}^B \quad (3)$$

where

$\hat{\underline{u}}_{Star}^B$ = Start tracker measured unit pointing vector to the star, transformed to the B frame.
$\hat{\underline{u}}_{Star}^I$ = Star tracker measurement in the I frame.

Comparison of $\hat{\underline{u}}_{Star}^I$ with the star catalog value $\underline{u}_{Star}^I$ then forms an error signal for Kalman filter input. Taken literally, the catalog value $\underline{u}_{Star}^I$ would be subtracted from $\hat{\underline{u}}_{Star}^I$ to form a three-component error signal. However, as will be apparent subsequently in (13), only the error components of $\hat{\underline{u}}_{Star}^I$ perpendicular to $\underline{u}_{Star}^I$ appear in $\hat{\underline{u}}_{Star}^I$. Hence, by subtracting $\underline{u}_{Star}^I$ from $\hat{\underline{u}}_{Star}^I$ to form a Kalman filter error input error, the three error components will not be independent, creating ill-conditioning in the Kalman gain calculation process. An alternative (described next) avoids the problem by explicitly calculating two orthogonal components of

$\underline{\hat{u}}^{\mathrm{I}}_{Star}$ perpendicular to $\underline{u}^{\mathrm{I}}_{Star}$, then using them to form a two-component Kalman filter input vector.

Using the true pointing vector to the star from the star catalog, a star tracking error signal is first formed as

$$\underline{e}^{\mathrm{I}}_{Star} = \underline{\hat{u}}^{\mathrm{I}}_{Star} \times \underline{u}^{\mathrm{I}}_{Star} \tag{4}$$

where

$\underline{u}^{\mathrm{I}}_{Star}$ = True unit vector pointing to the star in the I frame (from a star catalog).
$\underline{e}^{\mathrm{I}}_{Star}$ = Star tracking error vector in the I frame.

With (3), (4) becomes:

$$\underline{e}^{\mathrm{I}}_{Star} = \left(\hat{C}^{\mathrm{I}}_{\mathrm{B}} \, \underline{\hat{u}}^{B}_{Star} \right) \times \underline{u}^{\mathrm{I}}_{Star} \tag{5}$$

To find the two components of $\underline{e}^{\mathrm{I}}_{Star}$ perpendicular to $\underline{u}^{\mathrm{I}}_{Star}$, two mutually orthogonal unit vectors are formed, both perpendicular to $\underline{u}^{\mathrm{I}}_{Star}$, and their dot product taken with $\underline{e}^{\mathrm{I}}_{Star}$. For example, of the three unit vectors along I frame axes $\underline{i}^{\mathrm{I}}$, $\underline{j}^{\mathrm{I}}$, $\underline{k}^{\mathrm{I}}$, the one having the smallest dot product with $\underline{e}^{\mathrm{I}}_{Star}$ is first found; call it $\underline{p}^{\mathrm{I}}$. The desired orthogonal unit vectors ($\underline{u}^{\mathrm{I}}_{Star\perp 1}$ and $\underline{u}^{\mathrm{I}}_{Star\perp 2}$) perpendicular to $\underline{u}^{\mathrm{I}}_{Star}$ are then formed as

$$\underline{u}^{\mathrm{I}}_{Star\perp 1} = \underline{u}^{\mathrm{I}}_{Star} \times \underline{p}^{\mathrm{I}} / \left| \underline{u}^{\mathrm{I}}_{Star} \times \underline{p}^{\mathrm{I}} \right| \qquad \underline{u}^{\mathrm{I}}_{Star\perp 2} = \underline{u}^{\mathrm{I}}_{Star} \times \underline{u}_{Star\perp 1} \tag{6}$$

Finally, $u^{\mathrm{I}}_{Star\perp 1}$ and $u^{\mathrm{I}}_{Star\perp 2}$ are used to calculate the e^{I}_{Star} components perpendicular to $\underline{u}^{\mathrm{I}}_{Star}$ for Kalman filter observation input [2, Sect. 5.2]. Treating the components as elements of a column matrix vector, the result is

$$\underline{M} = \begin{bmatrix} \underline{u}^{\mathrm{I}}_{Star\perp 1} \cdot \underline{e}^{\mathrm{I}}_{Star} \\ \underline{u}^{\mathrm{I}}_{Star\perp 2} \cdot \underline{e}^{\mathrm{I}}_{Star} \end{bmatrix} = F^{\mathrm{I}}_{Star} \, \underline{e}^{\mathrm{I}}_{Star} \qquad F^{\mathrm{I}}_{Star} \equiv \begin{bmatrix} \left(\underline{u}^{\mathrm{I}}_{Star\perp 1} \right)^T \\ \left(\underline{u}^{\mathrm{I}}_{Star\perp 2} \right)^T \end{bmatrix} \tag{7}$$

where

$\underline{M}$ = Kalman filter two element observation input vector.

ERROR MODELS FOR KALMAN FILTER DESIGN

The Stellar Aiding Kalman filter is constructed based on linearized error models for parameters within $\underline{e}_{Star}^{\mathrm{I}}$ that impact the (7) observation, i.e., within $\hat{\mathrm{C}}_{\mathrm{B}}^{\mathrm{I}}$ and $\tilde{\underline{u}}_{Star}^{B}$ of $\underline{e}_{Star}^{\mathrm{I}}$ in (5). An analytical model for $\hat{\underline{u}}_{Star}^{B}$ can be defined assuming that its linearized error is caused by star-tracker misalignment to the B frame:

$$\hat{\underline{u}}_{Star}^{B} = \left[I - \left(\delta\underline{\alpha}_{StrTrkr} \times \right) \right] \underline{u}_{Star}^{\mathrm{B}} = \left[I - \left(\delta\underline{\alpha}_{StrTrkr} \times \right) \right] \mathrm{C}_{\mathrm{I}}^{\mathrm{B}} \, \underline{u}_{Star}^{\mathrm{I}}$$
$$\delta\dot{\underline{\alpha}}_{StrTrkr} = 0 \quad \rightarrow \quad \text{or maybe more complicated} \tag{8}$$

where

$\delta\underline{\alpha}_{StrTrkr}$ = Star-tracker misalignment error vector.

I = Identity matrix.

A model for $\hat{\mathrm{C}}_{\mathrm{B}}^{\mathrm{I}}$ error can be defined based on its linearized rotation angle error vector definition in [1, Eq. (46)]:

$$\left(\underline{\psi}^{\mathrm{I}} \times \right) = I - \hat{\mathrm{C}}_{\mathrm{B}}^{\mathrm{I}} \, \mathrm{C}_{\mathrm{I}}^{\mathrm{B}} + \left[\left(\mathrm{C}_{\mathrm{B}}^{\mathrm{I}} \, \underline{n}_{Quant} \right) \times \right] \tag{9}$$

where

$\underline{\psi}^{\mathrm{I}}$ = Small rotation angle error vector in $\hat{\mathrm{C}}_{\mathrm{B}}^{\mathrm{I}}$ projected on I frame axes.

$\underline{n}_{Quant}$ = Gyro output quantization noise vector.

Equation (9) rearranged obtains an analytical model for $\hat{\mathrm{C}}_{\mathrm{B}}^{\mathrm{I}}$:

$$\hat{\mathrm{C}}_{\mathrm{B}}^{\mathrm{I}} = \left\{ I - \left[\left(\underline{\psi}^{\mathrm{I}} - \mathrm{C}_{\mathrm{B}}^{\mathrm{I}} \, \underline{n}_{Quant} \right) \times \right] \right\} \mathrm{C}_{\mathrm{B}}^{\mathrm{I}} \tag{10}$$

The rate of change of $\underline{\psi}^{\mathrm{I}}$ is from [1, Eq. (51)]:

$$\dot{\underline{\psi}}^{\mathrm{I}} = -\,\mathrm{C}_{\mathrm{B}}^{\mathrm{I}} \left(\delta\underline{\omega}^{\mathrm{B}} - \underline{\omega}^{\mathrm{B}} \times \underline{n}_{Quant} \right) \approx -\,\mathrm{C}_{\mathrm{B}}^{\mathrm{I}} \, \delta\underline{\omega}^{\mathrm{B}} \tag{11}$$

in which from [1, Eq. (50)]:

$$
\begin{aligned}
\delta\underline{\omega}^{\mathrm{B}} &\equiv \underline{\hat{\omega}}^{\mathrm{B}} - \underline{\omega}^{\mathrm{B}} = \delta K_{Scal/Mis}\, \underline{\omega}^{\mathrm{B}} + \delta\underline{K}_{Bias} + \underline{n}_{RndWlk} \\
&= \Omega^{\mathrm{B}}\, \delta\underline{K}_{Scal/Mis} + \delta\underline{K}_{Bias} + \underline{n}_{RndWlk} \qquad (12) \\
\delta\dot{\underline{K}}_{Scal/Mis} &= 0 \qquad \delta\dot{\underline{K}}_{Bias} = 0 \quad \rightarrow \quad \text{Depending on gyro}
\end{aligned}
$$

where

$\delta K_{Scal/Mis}$ = Gyro scale-factor-error/misalignment error matrix.

$\delta\underline{K}_{Scal/Mis} = \delta K_{Scal/Mis}$ with its three columns arranged in a single 9 element column format.

Ω^{B} = Square 9 × 9 matrix containing elements of $\underline{\omega}^{\mathrm{B}}$ such that

$$\Omega^{\mathrm{B}}\, \underline{\delta K}_{Scal/Mis} = \delta K_{Scal/Mis}\, \underline{\omega}^{\mathrm{B}} .$$

$\delta\underline{K}_{Bias}$ = Gyro bias error vector.

$\underline{n}_{RndWlk}$ = Gyro random walk on attitude noise.

Based on the previous results, an error model for $\underline{e}^{\mathrm{I}}_{Star}$ can now be formed by substituting (10) and (8) into (5). For the $\hat{\mathrm{C}}^{\mathrm{I}}_{\mathrm{B}}\, \tilde{\underline{u}}^{\mathrm{B}}_{Star}$ term in (5) (defined in (3) as $\hat{\underline{u}}^{\mathrm{I}}_{Star}$):

$$
\begin{aligned}
\hat{\underline{u}}^{\mathrm{I}}_{Star} &= \hat{\mathrm{C}}^{\mathrm{I}}_{\mathrm{B}}\, \hat{\underline{u}}^{\mathrm{B}}_{Star} = \left\{ I - \left[\left(\underline{\psi}^{\mathrm{I}} - \mathrm{C}^{\mathrm{I}}_{\mathrm{B}}\, \underline{n}_{Quant} \right) \times \right] \right\} \mathrm{C}^{\mathrm{I}}_{\mathrm{B}} \left[I - \left(\delta\underline{\alpha}_{StrTrkr} \times \right) \right] \mathrm{C}^{\mathrm{B}}_{\mathrm{I}}\, \underline{u}^{\mathrm{I}}_{Star} \\
&= \left\{ I - \left[\left(\underline{\psi}^{\mathrm{I}} - \mathrm{C}^{\mathrm{I}}_{\mathrm{B}}\, n_{Quant} \right) \times \right] \right\} \left\{ I - \left[\left(\mathrm{C}^{\mathrm{I}}_{\mathrm{B}}\, \delta\underline{\alpha}_{StrTrkr} \right) \times \right] \right\} \underline{u}^{\mathrm{I}}_{Star} \\
&= \left\langle I - \left\{ \left[\underline{\psi}^{\mathrm{I}} + \mathrm{C}^{\mathrm{I}}_{\mathrm{B}} \left(\delta\underline{\alpha}_{StrTrkr} - \underline{n}_{Quant} \right) \right] \times \right\} \right\rangle \underline{u}^{\mathrm{I}}_{Star} \qquad (13) \\
&= \underline{u}^{\mathrm{I}}_{Star} - \left[\underline{\psi}^{\mathrm{I}} + \mathrm{C}^{\mathrm{I}}_{\mathrm{B}} \left(\delta\underline{\alpha}_{StrTrkr} - \underline{n}_{Quant} \right) \right] \times \underline{u}^{\mathrm{I}}_{Star}
\end{aligned}
$$

Substituting (13) into (5) then finds for a single star sighting:

$$
\begin{aligned}
\underline{e}^{\mathrm{I}}_{Star} &= - \left\{ \left[\underline{\psi}^{\mathrm{I}} + \mathrm{C}^{\mathrm{I}}_{\mathrm{B}} \left(\delta\underline{\alpha}_{StrTrkr} - \underline{n}_{Quant} \right) \right] \times \underline{u}^{\mathrm{I}}_{Star} \right\} \times \underline{u}^{\mathrm{I}}_{Star} \qquad (14) \\
&= \left[I - \underline{u}^{\mathrm{I}}_{Star} \left(\underline{u}^{\mathrm{I}}_{Star} \cdot \right) \right] \left[\underline{\psi}^{\mathrm{I}} + \mathrm{C}^{\mathrm{I}}_{\mathrm{B}} \left(\delta\underline{\alpha}_{StrTrkr} - \underline{n}_{Quant} \right) \right]
\end{aligned}
$$

The Kalman filter measurement vector $\underline{z}$ is the linearized form of the (7) two component observation vector $\underline{M}$ [2, Sect. 5.2]. Thus, substituting (14) in (7):

$$\underline{z} = F^{\mathrm{I}}_{Star} \left[I - \underline{u}^{\mathrm{I}}_{Star} \left(\underline{u}^{\mathrm{I}}_{Star} \cdot \right) \right] \left[\underline{\psi}^{\mathrm{I}} + \mathrm{C}^{\mathrm{I}}_{\mathrm{B}} \left(\delta\underline{\alpha}_{StrTrkr} - \underline{n}_{Quant} \right) \right] \qquad (15)$$

where

$\underline{z}$ = Measurement vector, the linearized form of observation vector $\underline{M}$.

Additional measurements would be made to other stars at different times to determine the three components of $\underline{\psi}^{\mathrm{I}}$, $\delta\underline{\alpha}_{StrTrkr}$ in (15), and the fixed gyro errors in (12). Note: Accurate determination of $\underline{\psi}^{\mathrm{I}}$ can only be made after $\delta\underline{\alpha}_{StrTrkr}$ has been estimated by the Kalman filter so that its presence in the observation vector can be properly accounted for during the $\underline{\psi}^{\mathrm{I}}$ estimation process. Sustained attitude accuracy following each stellar attitude correction can only be achieved after the fixed gyro errors in (12) have been accurately estimated/corrected to eliminate their error buildup in $\underline{\psi}^{\mathrm{I}}$ between stellar updates.

It is to be noted that (15) (the linearized form of observation vector $\underline{M}$ input (7) to the Kalman filter) is a direct measure of the $\underline{\psi}^{\mathrm{I}}$ attitude error parameter. For this reason, error parameter $\underline{\psi}^{\mathrm{I}}$ (also used extensively for inertial navigation system attitude error modeling in general, e.g., [3, Sect. 7.1]), has been commonly referred to as the "stellar pointing error".

MATRIX FORMATTING

Equations (8), (11), (12), and (15) can be formatted for matrix implementation in a standard Extended Kalman Filter [2, Sects. 5.1 and 5.2] as illustrated in Appendix A:

$$\begin{aligned} \dot{\underline{x}} &= A\,\underline{x} + G_P\,\underline{n}_P \quad \text{Error state dynamic equation} \\ \underline{z} &= H\,\underline{x} + G_M\,\underline{n}_M \quad \text{Measurement equation} \end{aligned} \tag{16}$$

where

$\underline{x}$ = Error state vector.
A = Error state dynamic matrix.
G_P = Process noise dynamic coupling matrix.
$\underline{n}_P$ = Vector of independent process noise elements.
$\underline{z}$ = Measurement vector.
H = Measurement matrix.
G_M = Measurement noise dynamic coupling matrix.
$\underline{n}_M$ = Vector of independent measurement noise elements.

The error state vector $\underline{x}$ for Stellar Aided INS Kalman filter design would be formed from $\underline{\psi}^{\mathrm{I}}, \delta\underline{\alpha}_{StrTrkr}, \delta\underline{K}_{Scal/Mis}, \delta\underline{K}_{Bias}$. With this definition, error model equations (8), (11), (12), and (15) become in matrix form:

$$\underline{x} \equiv \begin{bmatrix} \underline{\psi}^{\mathrm{I}} \\ \delta\underline{\alpha}_{StrTrkr} \\ \delta\underline{K}_{Scal/Mis} \\ \delta\underline{K}_{Bias} \end{bmatrix} \qquad \dot{\underline{x}} = \begin{bmatrix} 0 & 0 & -\mathrm{C}_{\mathrm{B}}^{\mathrm{I}}\,\Omega^{\mathrm{B}} & -\mathrm{C}_{\mathrm{B}}^{\mathrm{I}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} -\mathrm{C}_{\mathrm{B}}^{\mathrm{I}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \underline{n}_{RndWlk} \qquad (17)$$

$$\underline{z} = F_{Star}^{\mathrm{I}} \left[I - \underline{u}_{Star}^{\mathrm{I}} \left(\underline{u}_{Star}^{\mathrm{I}} \cdot \right) \right] \begin{bmatrix} \mathrm{I} & \mathrm{C}_{\mathrm{B}}^{\mathrm{I}} & 0 & 0 \end{bmatrix} \underline{x} - F_{Star}^{\mathrm{I}} \left[I - \underline{u}_{Star}^{\mathrm{I}} \left(\underline{u}_{Star}^{\mathrm{I}} \cdot \right) \right] \mathrm{C}_{\mathrm{B}}^{\mathrm{I}}\, \underline{n}_{Quant} \qquad (18)$$

Comparing (17) and (18) to the standard (16) form identifies the Kalman filter matrix design elements as

$$\underline{x} = \begin{bmatrix} \underline{\psi}^{\mathrm{I}} \\ \delta\underline{\alpha}_{StrTrkr} \\ \delta\underline{K}_{Scal/Mis} \\ \delta\underline{K}_{Bias} \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & -\mathrm{C}_{\mathrm{B}}^{\mathrm{I}}\,\Omega^{\mathrm{B}} & -\mathrm{C}_{\mathrm{B}}^{\mathrm{I}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad G_P = \begin{bmatrix} -\mathrm{C}_{\mathrm{B}}^{\mathrm{I}} \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{n}_P = \underline{n}_{RndWlk}$$

$$H = F_{Star}^{\mathrm{I}} \left[I - \underline{u}_{Star}^{\mathrm{I}} \left(\underline{u}_{Star}^{\mathrm{I}} \cdot \right) \right] \begin{bmatrix} \mathrm{I} & \mathrm{C}_{\mathrm{B}}^{\mathrm{I}} & 0 & 0 \end{bmatrix}$$
$$G_M = -F_{Star}^{\mathrm{I}} \left[I - \underline{u}_{Star}^{\mathrm{I}} \left(\underline{u}_{Star}^{\mathrm{I}} \cdot \right) \right] \mathrm{C}_{\mathrm{B}}^{\mathrm{I}} \qquad \underline{n}_M = \underline{n}_{Quant} \qquad (19)$$

In Appendix A, error state vector $\underline{x}$ and measurement vector $\underline{z}$ appear in the Fig. A-1 Kalman estimation/process diagram. Error state dynamic matrix A and measurement matrix H appear in Fig. A-1, and also in Appendix A Kalman gain determination equations (A-1). The process noise matrix G_P and measurement noise dynamic coupling matrix G_M also appear in (A-1) for the Kalman gains. Error state vector $\underline{x}$, process noise vector $\underline{n}_P$, and measurement noise vector $\underline{n}_M$ appear in (A-1) for gain calculations, but only in a statistical sense. The spectral densities of $\underline{n}_P$ appear as elements of diagonal process noise matrix Q_P. The $\underline{n}_M$ variances appear as the elements of diagonal measurement noise matrix R_M. The uncertainty $\underline{\tilde{x}}$ in the estimated error state vector $\underline{\hat{x}}$ appears in (A-1) as square covariance matrix P representing the expected value of $\underline{\tilde{x}}$ multiplied by its transpose: $\underline{\tilde{x}} \equiv \underline{\hat{x}} - \underline{x}$ and $P \equiv E\left(\underline{\tilde{x}}\,\underline{\tilde{x}}^T\right)$ where E ( ) is the expected value operator.

One final point (not identified in Appendix A), is the practice in some actual Kalman filter design configurations to constrain covariance matrix *P* to ever have elements becoming too small or high than is reasonably likely, considering uncertainties in the error models used for Kalman filter design. This is easily achieved by imposing upper and lower limits on *P* as part of Appendix A equations (A-1) Kalman covariance cyclic update operations [4, Sect. 15.1.2.1.1.4].

APPENDIX A

INERTIAL NAVIGATION SYSTEM KALMAN FILTER AIDING STRUCTURE

Kalman Filter aiding of an Inertial Navigation System (INS) (Fig. A-1) is a dynamic process in which INS computed navigation data is periodically compared with equivalent reference navigation data (at cycle rate n), and used in feedback fashion to update INS error parameters.

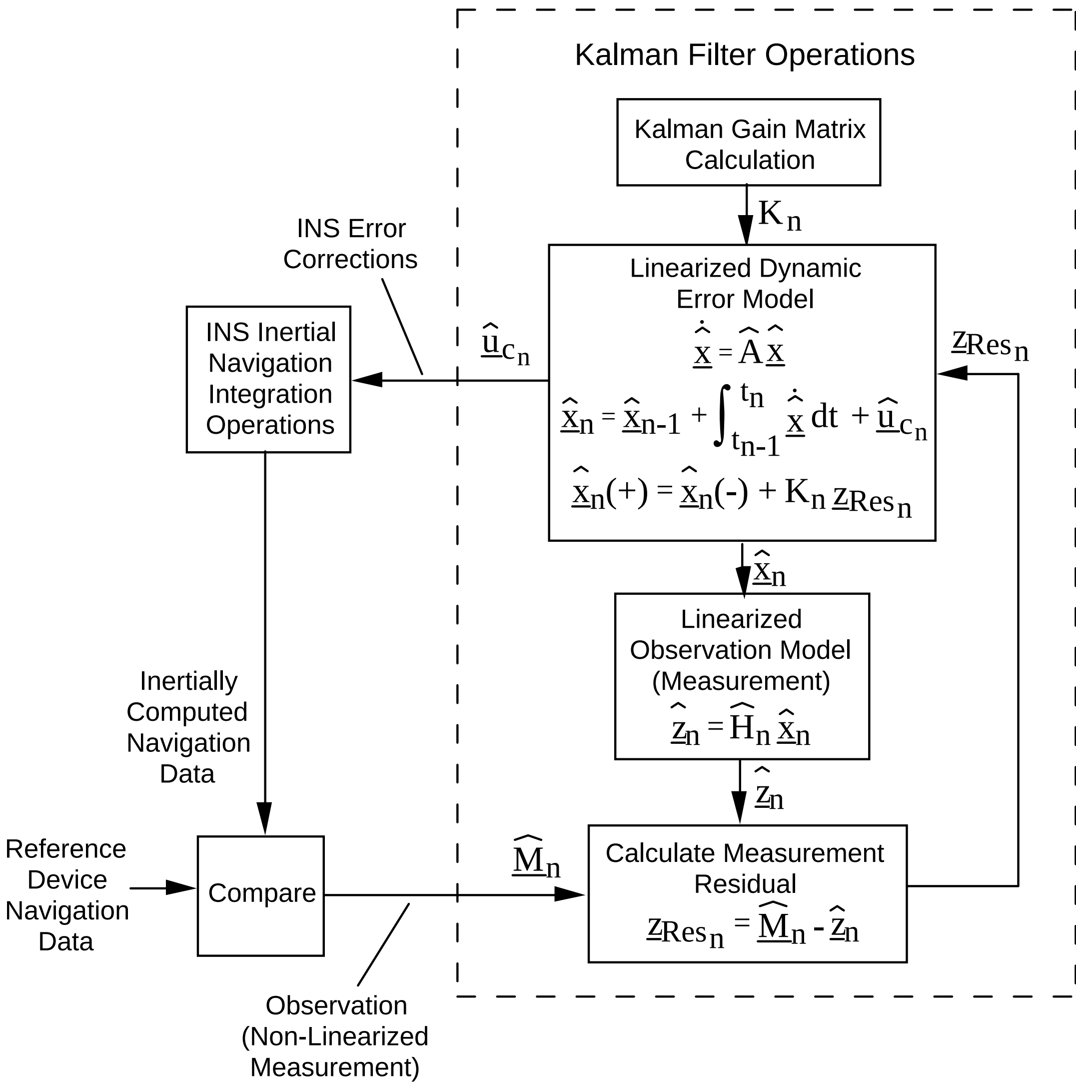

Fig. A-1: Kalman Filter Inertial Navigation Aiding

Note in Fig. A-1 that all parameters are shown with a $\widehat{(\ )}$ designation to indicate that they are computed estimates within the INS and reference navigation device of actual equivalent ( ) parameters. The analytic details of the Fig. A-1 operations are provided in [4 - Chapt. 15, 5, 6, 7 - pp. 415 - 457].

In Fig. A-1, the inertially-computed/reference-device navigation data comparison $\underline{\widehat{M}}$ ("observation vector") is input to the Kalman filter where it is compared against a linearized estimate of $\underline{\widehat{M}}$, "measurement vector" $\underline{\hat{z}}$. The equation for $\underline{\hat{z}}$ is based on estimates of expected errors (embodied in the error state vector column matrix $\underline{\hat{x}}$) generated by a linearized dynamic error model of inertial navigation and reference device operations, and how they couple into the measurement (through the "measurement matrix" $\widehat{H}$). (The error state dynamic matrix $\widehat{A}$ in Fig. A-1 defines the dynamics of how $\underline{\hat{x}}$ propagates from the last n cycle to the current n cycle.) The difference between the observation $\underline{\widehat{M}}$ and estimated measurement $\underline{\hat{z}}$ (the "measurement residual" $\underline{z}_{Res}$), is multiplied by a Kalman gain matrix K_n to generate corrections to the Kalman filter error estimates. The control vector $\underline{\hat{u}}_c$ formed from INS error estimates in $\underline{\hat{x}}$ (including provisions for $\underline{\hat{x}}$ computation delay) is used to correct the INS by subtraction from the equivalent INS parameter data. To account for the $\underline{\hat{u}}_c$ corrections applied to the INS, the $\underline{\hat{u}}_c$ vector is also used to update the Kalman $\underline{\hat{x}}$ error model for the applied INS error correction.

The K_n Kalman gain matrix in Fig. A-1 is computed at each n cycle from a statistical model of the expected uncertainty in the Fig. A-1 linearized updating process, a function of the error state covariance matrix P:

$$P_n = P_{n-1} + \int_{t_{n-1}}^{t_n} \dot{P}\, dt \qquad \dot{P} = \widehat{A}\, P + P\, \widehat{A}^T + \widehat{G}_P\, Q_P\, \widehat{G}_P^T$$
$$K_n = P_n(-)\, \widehat{H}_n^T \left(\widehat{H}_n\, P_n(-)\, \widehat{H}_n^T + \widehat{G}_{M_n}\, R_M\, \widehat{G}_{M_n}^T \right)^{-1} \qquad \text{(A-1)}$$
$$P_n(+) = \left(I - K_n\, \widehat{H}_n\right) P_n(-) \left(I - K_n\, \widehat{H}_n\right)^T + K_n\, \widehat{G}_{M_n}\, R_M\, \widehat{G}_{M_n}^T\, K_n^T$$

The P covariance in equations (A-1) is analytically defined as $E\left(\underline{x}_{Uncrtny}\, \underline{x}_{Uncrtny}^T\right)$ where E is the expected value operator and $\underline{x}_{Uncrnty}$ is the uncertainty in the error state estimate $\underline{\hat{x}}$ compared with the true value $\underline{x}$. The covariance matrix measures how initial uncertainties in $\underline{\hat{x}}$ (at the start of Kalman aiding) are progressively reduced by the Fig. A-1 dynamic estimation/updating process, and how unaccounted for noise effects (in $\underline{\hat{x}}$ propagation between updates and measurement updating) delay the convergence process. Noise parameters incorporated in (A-1) gain determination operations are the Q_P process noise matrix that accounts for random INS error buildup between n cycles, the $\widehat{G}_P$ matrix that couples the process noise into error state uncertainty components, the measurement noise matrix R_M that accounts for random errors in the observation and in calculation of the measurement residual, and the $\widehat{G}_M$ matrix that couples the measurement noise into the measurement residual components [4 - Sect. 15.1] and [7 - pp. 428].

The success of the Fig. A-1 process depends on the accuracy by which the Kalman filter linearized models match the actual operations in the INS and reference navigation device. An important element in this regard is the impact of linearization on the measurement residual. Second order components in the Fig. A-1 observation vector $\underline{M}$ are ignored in the Kalman filter linearized models, hence, will appear in the measurement residual $\hat{\underline{z}}_{Res}$ and modify $\underline{x}$ through the Kalman gains. Since the gains do not account for second order errors, the result will add unknown errors to $\hat{\underline{x}}$. To minimize the impact of second order errors on Kalman filter performance, it has been previous practice to assure that initial errors are small enough that second order residuals become negligible, e.g., [8]. However, in some applications, residual second order Kalman filter modeling errors can still produce error mis-estimation under particular dynamic conditions. Recent publications [9, 10] have developed methods to mitigate second order error effects in linear Kalman filters for some applications.

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