DOWN-SUMMING ROTATION VECTORS FOR STRAPDOWN ATTITUDE UPDATING

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WBN-14019
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July 16, 2017

ABSTRACT

This article defines a general analytical formula for combining a sequence of rotation vectors into a single rotation vector. The basic concept has been traditionally used in strapdown inertial navigation systems (INSs) for converting ("summing-down") rotation vectors calculated at high rate (to measure vibration induced high frequency angular motion) into a rotation vector for lower speed attitude updating. It is shown how the formula also has application to integrated digital systems in which high speed rotation vectors are generated in a strapdown inertial measurement unit (IMU), then output to a separate system computer for lower rate attitude updating.

INTRODUCTION

Attitude (angular orientation) in a modern-day strapdown inertial navigation system (INS) is updated at a prescribed rate using a rotation vector. For attitude represented as a direction cosine matrix, the operation would be as follows [4, 6, 10]:

$$C_n = C_{n-1}\left[I + f_1\left(\underline{\phi}_n \times\right) + f_2\left(\underline{\phi}_n \times\right)^2 \right]$$
$$f_{1_n} \equiv \frac{\sin\phi_n}{\phi_n} = 1 - \frac{\phi_n^2}{3!} + \frac{\phi_n^4}{5!} + \cdots \qquad f_{2_n} \equiv \frac{1-\cos\phi_n}{\phi^2} = 1 - \frac{\phi_n^2}{2!} + \frac{\phi_n^4}{4!} + \cdots \tag{1}$$

where

n = Attitude update cycle index.

C_n, C_{n-1} = Direction cosine matrix at the n and n-1 attitude update cycle times.

$\underline{\phi}_n$ = Rotation vector.

ϕ_n = Magnitude of $\underline{\phi}_n$.

$\left(\underline{\phi}_n \times\right)$ = Cross-product operator form of $\underline{\phi}_n$ defined such that when formatted as a square matrix, its product with an arbitrary column-matrix formatted vector equals the cross-product of $\underline{\phi}_n$ with that vector, i.e., $\left(\underline{\phi}_n \times\right) \underline{V} = \underline{\phi}_n \times \underline{V}$.

The equivalent to the (1) operation is also commonly performed using $\underline{\phi}_n$ for attitude quaternion updating [13].

The rotation vector $\underline{\phi}_n$ in (1) measures the change in angular orientation over the *n*-1 to *n* attitude update time interval and is calculated for INS application as an integration process using angular rotation rate measurements from an orthogonal set of strapdown gyros. The rotation vector is analytically formed by calculating a "coning correction", then adding it to the integrated gyro sensed angular rate over the attitude update time interval.

In a modern-day strapdown digital INS, a two-speed architecture is commonly used for attitude updating The general concept was originated by the author in 1966 as a means for reducing computer throughput [3]. With the two-speed approach, attitude is updated at a basic computation speed using inputs generated by a high speed algorithm designed to accurately measure high frequency dynamic angular rotation effects (coning motion). The original two-speed concept was formulated as a truncated Picard expansion of integrated attitude change rate. The concept was refined in 1968 by Jordan [4] by defining the high speed operation as a rotation vector computation based on a linearized version of the Goodman-Robinson theorem [2]. In place of the truncated Picard expansion formula, attitude updating in [4] also incorporated an exact Euler attitude change formula (originally adopted in 1949 by Laning [1] for future strapdown inertial application). Subsequent two-speed formulations have used the Jordan framework, but with the rotation vector defined as an integrated approximation to the exact rotation vector rate equation originated by Laning in [1]. (The simplified Laning rotation vector rate equation is analytically identical to that derived by Jordan in [4] based on [2].) Since 1969, further work on attitude updating has focused on sophisticated versions of the coning correction vector to accurately measure dynamic angular rate in vibration and maneuvering environments, e.g., [5, 7 - 8, 11-12].

Inherent in the design of two-speed algorithms must be a mechanism for converting high-speed calculated coning corrections into a rotation vector suitable for lower speed attitude updating (a "summing-down" type of operation). The basic sum-down concept has been imbedded in many attitude algorithm articles without specific analytical identification, e.g., [4, 6 - 11]. The purpose of this article is to isolate the conversion operation as a general process applicable to attitude updating in general. The analytical approach defines a generalized formula for combining a sequence of rotation vectors (each representing attitude change over a short time interval) into a single rotation vector representing attitude change over a longer time interval (e.g., for attitude updating).

BASIC ROTATION VECTOR ANALYTICS

The rate of change of a rotation vector is given by the classical Lanning equation [1]:

$$\begin{aligned}\dot{\underline{\phi}} &= \underline{\omega} + \frac{1}{2}\underline{\phi}\times\underline{\omega}\times + \frac{1}{\phi^2}\left(1 - \frac{\phi\sin\phi}{2(1-\cos\phi)}\right)\underline{\phi}\times\left(\underline{\phi}\times\underline{\omega}\right) \\ &= \underline{\omega} + \frac{1}{2}\underline{\phi}\times\underline{\omega}\times + \frac{1}{12}\left(1 + \frac{1}{60}\phi^2 + \cdots\right)\underline{\phi}\times\left(\underline{\phi}\times\underline{\omega}\right)\end{aligned} \tag{1}$$

where

$\underline{\phi}$ = Rotation vector that measures angular rotation from some reference time point.

$\underline{\omega}$ = Angular rate vector that would be measured by a strapdown gyro triad.

The approximation to (1) commonly used in practice [4, 6 - 12] is

$$\dot{\underline{\phi}} \approx \underline{\omega} + \frac{1}{2}\underline{\alpha}\times\underline{\omega} \qquad \dot{\underline{\alpha}} \equiv \underline{\omega} \tag{2}$$

The integral of (2) over a particular time interval provides the associated rotation vector for that time period:

$$\underline{\phi}_l = \underline{\alpha}_l + \delta\underline{\phi}_l \qquad \underline{\alpha}(t) \equiv \int_{t_{l-1}}^{t}\underline{\omega}\,dt \qquad \delta\underline{\phi}_l \equiv \frac{1}{2}\int_{t_{l-1}}^{t_l}\underline{\alpha}(t)\times\underline{\omega}\,dt \qquad \underline{\alpha}_l \equiv \underline{\alpha}(t_n) = \int_{t_{l-1}}^{t_l}\underline{\omega}\,dt \tag{3}$$

where for the time period between arbitrary *l* computer cycles:

l = Computer cycle index for calculating $\underline{\phi}_l$.

$\underline{\alpha}_l$ = Integrated angular rate increment over the *l* cycle.

$\underline{\phi}_l$ = Rotation vector that measures angular rotation over the *l* cycle (i.e., from cycle time t_{l-1} to t_l).

$\delta\underline{\phi}_l$ = Commonly called the “coning correction” to integrated angular rate increment $\underline{\alpha}_l$.

A rotation vector can also be defined that measures angular rotation over an attitude updating time interval:

$$\underline{\phi}_n = \underline{\alpha}_n + \delta\underline{\phi}_n \qquad \underline{\alpha}(t) \equiv \int_{t_{n-1}}^{t}\underline{\omega}\,dt \qquad \delta\underline{\phi}_n \equiv \frac{1}{2}\int_{t_{n-1}}^{t_n}\underline{\alpha}(t)\times\underline{\omega}\,dt \qquad \underline{\alpha}_n \equiv \underline{\alpha}(t_n) = \int_{t_{n-1}}^{t_n}\underline{\omega}\,dt \tag{4}$$

where

n = Attitude updating computer cycle index.

$\underline{\phi}_n$ = Rotation vector that measures angular rotation between attitude update computer cycles n-1 and n.

$\underline{\alpha}_n$ = Integrated angular rate increment over the n cycle.

$\delta\underline{\phi}_n$ = The "coning correction" vector used in calculating $\underline{\phi}_n$, the correction to integrated angular rate increment $\underline{\alpha}_n$.

This article derives the algorithm for converting a sequence of $\underline{\phi}_l$ rotation vectors generated with (3) at computation rate l, into a single rotation vector spanning a longer n cycle time interval (a "summing-down" type operation). The result is identical to what would have been generated directly using (4).

ROTATION VECTOR SUM-DOWN CONVERSION ROUTINE

The rotation vector sum-down routine derives from an expanded form of (4) with a redefined version of (3):

$$\begin{gathered}
\underline{\alpha}(t) \equiv \int_{t_{n-1}}^{t} \underline{\omega}\, dt = \int_{t_{n-1}}^{t_{l-1}} \underline{\omega}\, dt + \int_{t_{l-1}}^{t} \underline{\omega}\, dt = \underline{\alpha}_{l-1} + \Delta\underline{\alpha}(t) \\
\Delta\underline{\alpha}(t) \equiv \int_{t_{l-1}}^{t} \underline{\omega}\, dt \qquad \Delta\underline{\alpha}_l \equiv \Delta\underline{\alpha}(t_l) = \int_{t_{l-1}}^{t_l} \underline{\omega}\, dt \\
\underline{\alpha}_{l-1} \equiv \int_{t_{n-1}}^{t_{l-1}} \underline{\omega}\, dt = \sum_{t_{n-1}}^{t_{l-1}} \Delta\underline{\alpha}_l \qquad \underline{\alpha}_n \equiv \int_{t_{n-1}}^{t_n} \underline{\omega}\, dt = \sum_{t_{n-1}}^{t_n} \Delta\underline{\alpha}_l
\end{gathered} \tag{5}$$

$$\underline{\phi}_l = \Delta\underline{\alpha}_l + \delta\underline{\phi}_l \qquad \delta\underline{\phi}_l \equiv \frac{1}{2}\int_{t_{l-1}}^{t_l} \Delta\underline{\alpha}(t) \times \underline{\omega}\, dt \tag{6}$$

$$\underline{\phi}_n = \underline{\alpha}_n + \delta\underline{\phi}_n \qquad \delta\underline{\phi}_n = \frac{1}{2}\int_{t_{n-1}}^{t_n} \underline{\alpha}(t) \times \underline{\omega}\, dt = \frac{1}{2}\sum_{t_{n-1}}^{t_n}\int_{t_{l-1}}^{t_l} \underline{\alpha}(t) \times \underline{\omega}\, dt \tag{7}$$

Substituting from (5) and (6) obtains for $\delta\underline{\phi}_n$ in (7):

$$\begin{aligned}
\delta\underline{\phi}_n &= \frac{1}{2}\sum_{t_{n-1}}^{t_n}\int_{t_{l-1}}^{t_l} \underline{\alpha}(t) \times \underline{\omega}\, dt = \frac{1}{2}\sum_{t_{n-1}}^{t_n}\int_{t_{l-1}}^{t_l} \left[\underline{\alpha}_{l-1} + \Delta\underline{\alpha}(t)\right] \times \underline{\omega}\, dt \\
&= \frac{1}{2}\sum_{t_{n-1}}^{t_n} \underline{\alpha}_{l-1} \times \int_{t_{l-1}}^{t_l} \underline{\omega}\, dt + \frac{1}{2}\sum_{t_{n-1}}^{t_n}\int_{t_{l-1}}^{t_l} \Delta\underline{\alpha}(t) \times \underline{\omega}\, dt \\
&= \frac{1}{2}\sum_{t_{n-1}}^{t_n} \underline{\alpha}_{l-1} \times \Delta\underline{\alpha}_l + \sum_{t_{n-1}}^{t_n} \delta\underline{\phi}_l
\end{aligned} \tag{8}$$

Thus, for $\underline{\phi}_n$ in (7) with (5) for $\underline{\alpha}_n$:

$$\underline{\phi}_n = \underline{\alpha}_n + \delta\underline{\phi}_n = \sum_{t_{n-1}}^{t_n} \Delta\underline{\alpha}_l + \frac{1}{2}\sum_{t_{n-1}}^{t_n} \underline{\alpha}_{l-1} \times \Delta\underline{\alpha}_l + \sum_{t_{n-1}}^{t_n} \delta\underline{\phi}_l$$
$$= \sum_{t_{n-1}}^{t_n} \left(\Delta\underline{\alpha}_l + \delta\underline{\phi}_l\right) + \frac{1}{2}\sum_{t_{n-1}}^{t_n} \underline{\alpha}_{l-1} \times \Delta\underline{\alpha}_l \qquad (9)$$

Identifying $\Delta\underline{\alpha}_l + \delta\underline{\phi}_l$ from (6) as rotation vector $\underline{\phi}_l$ and substituting $\underline{\alpha}_{l-1}$ from (5), Eq. (9) then obtains the formula for rotation vector sum-down conversion:

$$\underline{\alpha}_{l-1} = \sum_{t_{n-1}}^{t_{l-1}} \Delta\underline{\alpha}_l \qquad \underline{\phi}_n = \sum_{t_{n-1}}^{t_n} \left(\underline{\phi}_l + \frac{1}{2}\underline{\alpha}_{l-1} \times \Delta\underline{\alpha}_l\right) \qquad (10)$$

Eq. (10 is a general expression showing how rotation vectors $\underline{\phi}_l$ calculated at computation cycle rate *l* can be “summed-down” into a single rotation vector $\underline{\phi}_n$ covering a slower *n* cycle time interval.

AN INTERESTING APPLICATION

In addition to the traditional sum-down operation employed in a strapdown INS, an interesting new application for (10) is within an integrated digital system containing a strapdown inertial measurement unit (IMU) and a separate system computer in which inertial attitude updating and other navigation operations are performed. The IMU would contain the strapdown gyros with which rotation vector samples would be calculated at high speed as in (3). The IMU computed rotation vector sequence would then be output to the system computer for attitude updating at a slower computation rate. Conversion routine (10) would be used in the system computer to sum-down the IMU high rate rotation vector outputs into the equivalent rotation vector required for lower rate attitude updating.

CONCLUSION

This section hereby concludes the article.

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