LEVER ARM CORRECTIONS DURING INS TRANSFER ALIGNMENT WITH WIDE ANGLE INITIAL HEADING ERROR

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April 17, 2015 (Updated 09/15/16 - See last page for changes)

ABSTRACT

A recent article developed the methodology for Kalman filter alignment of an inertial navigation system (INS) with unknown initial heading under dynamic trajectory conditions. Reference data provided to the Kalman alignment process was GPS velocity with no heading input. To simplify analytics, the article assumed collocation of the INS and GPS antenna. This article expands on the wide heading alignment concept to account for lever arm displacement between the INS and GPS antenna using two alternative Kalman input measurement approaches; velocity matching (as in the original article), and integrated velocity matching. Included is a detailed discussion of transitioning from Kalman wide-heading-angle alignment into free-inertial or aided-inertial navigation.

INTRODUCTION

To assure accuracy, traditional linearized Kalman filters generally require operation under small system error conditions to minimize second order errors effects. When applied to INS initial alignment, this has required that initial three-axis attitude errors be small, necessitating a "Coarse Alignment" attitude initialization process before Kalman filter aided alignment can be initiated. A recent article [6] has shown how the input measurement to an INS Kalman alignment filter can be structured for minimum linearization error under large initial heading angle conditions. Using this approach, attitude initialization only requires "Coarse Leveling" to a small horizontal attitude error prior to Kalman alignment engagement. Reference [7] shows how rapid Coarse Leveling can be achieved in an INS using velocity data provided by a GPS receiver (or equivalent).

The [6] Kalman alignment article was based on forming the measurement with velocity data provided by a separate navigation reference device (e.g., a GPS receiver). To simplify the analytical development, [6] assumed that the INS and navigation reference were collocated in the user vehicle. In practice, physical separation between the INS and navigation reference device must be included in the Kalman alignment filter formulation to account for differences in INS and reference device velocity under vehicle angular rate (the so-called "lever arm" effect). (Note - Lever arm compensation is included in the [7] Coarse Leveling method.) This article expands on the [6] Kalman alignment approach to include lever arm compensation for two different measurement configurations: "velocity matching" based on a velocity comparison measurement (as in [6]), and "integrated velocity matching" based on a velocity integral comparison for the measurement. A comparison between the alternative measurement approaches is provided, demonstrating the advantages of integrated velocity matching in modeling lever arm effects under dynamic motion, including random flexure. This article also shows how uncertainty in GPS reference data can be incorporated within the Kalman alignment filter structure.

MATHEMATICAL NOTATION

The mathematical notation used in this article is the same as in references [1] - [7]:

$\underline{V}$ = Vector without specific coordinate frame designation. A vector is a parameter that has length and direction. Vectors used in this article are classified as "free vectors", hence, have no preferred location in coordinate frames in which they are analytically described.

$|\underline{V}|$ = Magnitude of vector $\underline{V}$.

$\underline{V}^A$ = Column matrix with elements equal to the projection of $\underline{V}$ on coordinate frame A axes. The projection of $\underline{V}$ on each frame A axis equals the dot product of $\underline{V}$ with a unit vector parallel to that coordinate axis.

$\left(\underline{V}^A \times\right)$ = Skew symmetric (or cross-product) form of $\underline{V}^A$ represented by the square matrix $\begin{bmatrix} 0 & -V_{ZA} & V_{YA} \\ V_{ZA} & 0 & -V_{XA} \\ -V_{YA} & V_{XA} & 0 \end{bmatrix}$ in which V_{XA}, V_{YA}, V_{ZA} are the components of $\underline{V}^A$. The matrix product of $\left(\underline{V}^A \times\right)$ with another A frame vector equals the cross-product of $\underline{V}^A$ with the vector in the A frame, i.e.: $\left(\underline{V}^A \times\right) \underline{W}^A = \underline{V}^A \times \underline{W}^A$.

C_{A2}^{A1} = Direction cosine matrix that transforms a vector from its coordinate frame $A2$ projection form to its coordinate frame $A1$ projection form, i.e.: $\underline{V}^{A1} = C_{A2}^{A1} \, \underline{V}^{A2}$. The columns of C_{A2}^{A1} are projections on $A1$ axes of unit vectors parallel to $A2$ axes. Conversely, the rows of C_{A2}^{A1} are projections on $A2$ axes of unit vectors parallel to $A1$ axes. An important property of C_{A2}^{A1} is that its inverse equals its transpose.

$\dot{(\;)} = \dfrac{d(\;)}{dt}$ = Time derivative of ( ).

$\underline{\omega}_{A1A2}$ = Angular rate vector of coordinate frame $A2$ relative to coordinate frame $A1$.

COORDINATE FRAMES

N = Locally level navigation coordinate frame (with Z axis up) used for attitude referencing and velocity/position integration operations in the INS. By definition, in this article, the initial heading of the N Frame is assumed to be nominal, i.e., error-free. Initial heading alignment of the N Frame relative to another known reference frame (N^*) is accounted for by defining the N frame to be nominally misaligned in heading from the N^* frame. It is further assumed that the N and N^* frames are rotated at the same angular rate so that the heading angle (around the Z up axis) between the two remains constant. Appendix A shows how this stipulation can be removed using a heading angle correction.

N^* = Locally level navigation coordinate frame (with Z axis up) used by the reference navigation device to deliver position/velocity data to the INS being aligned. The potentially large heading angle misalignment between the N and N^* frames is the means to account for initial heading error in the INS attitude data at the start of alignment. As defined, the Z axis of the N^* frame is parallel to the Z axis of the N Frame.

B = Strapdown inertial sensor coordinates ("body frame") with axes parallel to nominal right handed orthogonal INS sensor input axes.

I = Non-rotating inertial coordinate frame used as a symbolic reference for INS gyro angular rotation rate measurements as defined in [4].

E = Coordinate (earth) frame aligned with axes fixed to the earth (e.g., one axis parallel to earth's rotation axis, the other two axes parallel to earth's equatorial plane).

GENERAL KALMAN ALIGNMENT FILTER STRUCTURE

Kalman alignment is a specialized application of Kalman filter inertial aiding, a dynamic process in which INS computed navigation data is periodically compared with equivalent reference data (at cycle rate n), and used in feedback fashion to update (i.e., correct) INS navigation parameters. Equations (1) summarize the general updating process:

$$\begin{gathered}
\underline{\widehat{M}}_n = f\left(\xi_{INS_n}, \xi_{Ref_n}\right) \\
\underline{\hat{z}}_n = \widehat{H}_n\, \underline{\hat{x}}_n(-) \\
\underline{z}_{Res_n} = \underline{\widehat{M}}_n - \underline{\hat{z}}_n \\
\underline{\hat{x}}_n(+) = \underline{\hat{x}}_n(-) + K_n\, \underline{z}_{Res_n} \\
\underline{\hat{u}}_{c_n} = f\left(\underline{\hat{x}}_n(+)\right) \quad \text{Applied To } \xi_{INS} \text{ For Correction} \\
\underline{\hat{x}}_{n+1}(-) = \underline{\hat{x}}_n(+) + \int_{t_n}^{t_{n+1}} \dot{\underline{\hat{x}}}\, dt + \underline{\hat{u}}_{c_n} \qquad \dot{\underline{\hat{x}}} = \widehat{A}\, \underline{\hat{x}}
\end{gathered} \tag{1}$$

Parameters shown with a $\widehat{(\;)}$ designation in (1) identify computed estimates within the INS and reference navigation device of actual equivalent ( ) parameters. The (-) designation in (1) refers to the parameter value at the current n cycle, before it is updated (i.e., corrected) by the Kalman filter; the (+) designation refers to the parameter value at the current n cycle after Kalman updating. A summary description of the (1) operations is provided next. The analytic details are provided in [1 - pp. 415 - 486], [2 - Chapt. 15], [5], and [6].

In (1), data computed from INS navigation parameters ξ_{INS} are compared against equivalent data from the navigation reference ξ_{Ref} to form "observation" vector $\widehat{\underline{M}}$. Observation $\widehat{\underline{M}}$ is input to the Kalman filter where at each Kalman updating cycle n, it is compared against "measurement" $\hat{\underline{z}}$, a linearized estimate of $\widehat{\underline{M}}$. The equation for $\hat{\underline{z}}$ is based on linearized estimates of expected system errors (embodied in the error state vector column matrix $\hat{\underline{x}}$), and how they couple into measurement $\hat{\underline{z}}$ through "measurement matrix" $\widehat{H}$. The error state dynamic matrix $\widehat{A}$ in (1) defines the dynamics of how $\hat{\underline{x}}$ "propagates" from the last n cycle to the current n cycle.

The difference between observation $\widehat{\underline{M}}$ and estimated measurement $\hat{\underline{z}}$ (the "measurement residual" $\underline{z}_{Res}$), is multiplied by Kalman gain matrix K to generate corrections to the Kalman filter error estimates. The control vector $\hat{\underline{u}}_c$ formed from INS error estimates $\hat{\underline{x}}$ (including provisions for $\hat{\underline{x}}$ computation delay) is used to correct the INS data by subtraction from the equivalent INS parameters. To account for $\hat{\underline{u}}_c$ corrections applied to the INS, the $\hat{\underline{u}}_c$ vector is also used to update the Kalman filter $\hat{\underline{x}}$ error model for the applied INS error correction.

Kalman gain matrix K in (1) is computed at each n cycle with a statistical model of the expected uncertainty in the (1) linearized updating process, a function of the error state covariance matrix P:

$$K_n = P_n(-)\,\widehat{H}_n^T\left(\widehat{H}_n\,P_n(-)\,\widehat{H}_n^T + \widehat{G}_{M_n}\,R_M\,\widehat{G}_{M_n}^T\right)^{-1}$$

$$P_n(+) = \left(I - K_n\,\widehat{H}_n\right)P_n(-)\left(I - K_n\,\widehat{H}_n\right)^T + K_n\,\widehat{G}_{M_n}\,R_M\,\widehat{G}_{M_n}^T\,K_n^T \qquad (2)$$

$$P_{n+1}(-) = P_n(+) + \int_{t_n}^{t_{n+1}} \dot{P}\,dt \qquad \dot{P} = \widehat{A}\;P + P\,\widehat{A}^T + \widehat{G}_P\,Q_P\,\widehat{G}_P^T$$

where I is the identity matrix. The P covariance is analytically defined as $\mathcal{E}\left(\underline{\chi}\,\underline{\chi}^T\right)$ where $\mathcal{E}$ is the expected value operator and $\underline{\chi}$ is the uncertainty in the error state estimate $\hat{\underline{x}}$ compared with the $\underline{x}$ true value. The covariance matrix measures how the initial uncertainty in $\hat{\underline{x}}$ (at the start of Kalman alignment) is progressively reduced by the (1) dynamic estimation/updating process, and how unaccounted noise effects (in $\hat{\underline{x}}$ propagation between updates and $\underline{z}_{Res}$ measurement updating) delay and limit the convergence process. Noise parameters incorporated in the (2) gain determination operations are the Q_P process noise matrix that accounts for random INS error buildup in ξ_{INS} and ξ_{INS} between n cycles, the $\widehat{G}_P$ matrix that couples the process noise into error state uncertainty components, the measurement noise matrix R_M that accounts for random errors in the observation and measurement residual, and the $\widehat{G}_M$ matrix that couples measurement noise into the measurement residual components, [2 - Sect. 15.1] and [1 - pp. 428].

The remainder of this article will refer to equations (1) and (2), showing how derived velocity and integrated velocity alignment process equations fit into the general Kalman alignment structure.

ANALYTICS OF LEVER ARM COMPENSATION

The analytical basis for lever arm compensation derives from the positioning equation:

$$\underline{R}_{INS} = \underline{R}_{Ref} + \underline{l} \tag{3}$$

where

$\underline{R}$ = Position vector from earth's center to a designated location.

INS = Subscript identifying the parameter value at the INS location.

Ref = Subscript identifying the parameter value at the reference navigation device location.

$\underline{l}$ = Distance vector ("lever arm") from the reference navigation device to the INS.

The projection of (3) on *E* frame axes is:

$$\underline{R}_{INS}^{E} = \underline{R}_{Ref}^{E} + \underline{l}^{E} \tag{4}$$

The time rate of change of (4) defines the relationship between velocities relative to the earth at the INS and reference navigation device locations:

$$\underline{v}_{INS}^{E} = \underline{v}_{Ref}^{E} + \dot{\underline{l}}^{E} \qquad \underline{v}_{INS}^{E} \equiv \dot{\underline{R}}_{INS}^{E} \qquad \underline{v}_{Ref}^{E} \equiv \dot{\underline{R}}_{Ref}^{E} \tag{5}$$

where

$\underline{v}$ = Velocity relative to the earth.

Transformed to *N* frame axes, (5) becomes

$$\underline{v}_{INS}^{N} = \underline{v}_{Ref}^{N} + C_E^N \, \dot{\underline{l}}^E \tag{6}$$

The velocity output from the reference device is provided in *N** coordinates, hence, $\underline{v}_{Ref}^{N}$ in (6) can be expressed as:

$$\underline{v}_{Ref}^{N} = C_{N*}^{N} \, \underline{v}_{Ref}^{N*} \tag{7}$$

where

C_{N*}^{N} = Direction cosine matrix between the *N** and *N* frames which, by virtue of the frame definitions, represents a transformation operation around the local vertical, and is constant because both *N** and *N* are rotated at the same angular rate around the local vertical.

The $C_E^N \, \dot{\underline{l}}^E$ term in (6) can be expressed in terms of *B* frame components through the following development:

$$\begin{aligned}
\underline{l}^E &= C_B^E \, \underline{l}^B \\
\dot{\underline{l}}^E &= \dot{C}_B^E \, \underline{l}^B + C_B^E \, \dot{\underline{l}}^B \\
C_E^N \, \dot{\underline{l}}^E &= C_E^N \, \dot{C}_B^E \, \underline{l}^B + C_B^N \, \dot{\underline{l}}^B
\end{aligned} \tag{8}$$

But,

$$\dot{C}_B^N = \frac{d}{dt}\left(C_E^N \, C_B^E\right) = \dot{C}_E^N \, C_B^E + C_E^N \, \dot{C}_B^E \tag{9}$$

hence:

$$C_E^N \, \dot{C}_B^E = \dot{C}_B^N - \dot{C}_E^N \, C_B^E \tag{10}$$

With (10), the last equation in (8) becomes

$$C_E^N \, \dot{\underline{l}}^E = \left(\dot{C}_B^N - \dot{C}_E^N \, C_B^E\right) \underline{l}^B + C_B^N \, \dot{\underline{l}}^B \tag{11}$$

Further development of (11) depends on whether the measurement approach is velocity matching or integrated velocity matching.

<u>Lever Arm Effects For Velocity Matching Alignment</u>

For a velocity matching Kalman alignment approach, the $\dot{C}_B^N$ and $\dot{C}_E^N$ terms in (11) are from [2 - Eq. (3.3.2-13)] and the transpose of [2 - Eq. (3.3.2-6)]:

$$\dot{C}_B^N = C_B^N \left(\underline{\omega}_{IB}^B \times \right) - \left(\underline{\omega}_{IN}^N \times \right) C_B^N \qquad \dot{C}_E^N = - \left(\underline{\omega}_{EN}^N \times \right) C_E^N \tag{12}$$

in which it has been recognized that the transpose of C_N^E equals C_E^N, the transpose of the general skew symmetric form $\left(\underline{V}^A \times \right)$ equals its negative, and where

$\underline{\omega}_{IB}^B$ = Angular rate of the *B* frame relative to non-rotating inertial space (*I*) projected on *B* frame axes; the angular rate measured by the INS strapdown gyros.

$\underline{\omega}_{EN}^N$ = Angular rate of the *N* frame relative to the *E* frame projected on *N* frame axes.

$\underline{\omega}_{IN}^N$ = Angular rate of the *N* frame relative to non-rotating inertial space (*I*) projected on *N* frame axes; equal to the sum of $\underline{\omega}_{EN}^N$ plus $\underline{\omega}_{IE}^N$, earth's angular rate relative to inertial space.

Combining equations (12) into the (11) bracketed term with $\underline{\omega}_{IN}^N = \underline{\omega}_{IE}^N + \underline{\omega}_{EN}^N$ finds:

$$\begin{aligned}
\dot{C}_B^N - \dot{C}_E^N \, C_B^E &= C_B^N \left(\underline{\omega}_{IB}^B \times \right) - \left(\underline{\omega}_{IN}^N \times \right) C_B^N + \left(\underline{\omega}_{EN}^N \times \right) C_B^N = C_B^N \left(\underline{\omega}_{IB}^B \times \right) - \left(\underline{\omega}_{IE}^N \times \right) C_B^N \\
&= C_B^N \left(\underline{\omega}_{IB}^B \times \right) - C_B^N \left(C_B^N \right)^T \left(\underline{\omega}_{IE}^N \times \right) C_B^N = C_B^N \left(\underline{\omega}_{IB}^B \times \right) - C_B^N \left\{ \left[\left(C_B^N \right)^T \underline{\omega}_{IE}^N \right] \times \right\} \qquad (13) \\
&= C_B^N \left\{ \left[\underline{\omega}_{IB}^B - \left(C_B^N \right)^T \underline{\omega}_{IE}^N \right] \times \right\}
\end{aligned}$$

Substituting (13) in (11) yields:

$$C_E^N \, \dot{\underline{l}}^E = C_B^N \left\{ \left[\underline{\omega}_{IB}^B - \left(C_B^N \right)^T \underline{\omega}_{IE}^N \right] \times \underline{l}^B + \dot{\underline{l}}^B \right\} \tag{14}$$

With (7) and (14), (6) then becomes:

$$\underline{v}_{INS}^N = C_{N*}^N \, \underline{v}_{Ref}^{N*} + C_B^N \left\{ \left[\underline{\omega}_{IB}^B - \left(C_B^N \right)^T \underline{\omega}_{IE}^N \right] \times \underline{l}^B + \dot{\underline{l}}^B \right\} \tag{15}$$

Kalman alignment filter design using a velocity matching measurement is based on (15).

<u>Lever Arm Effects For Integrated Velocity Matching Alignment</u>

For an integrated velocity matching Kalman alignment approach, the equivalent to (15) is found from (11) with $\dot{C}_E^N$ from (12):

$$
\begin{aligned}
C_E^N \, \dot{\underline{l}}^E &= \left(\dot{C}_B^N - \dot{C}_E^N \; C_B^E \right) \underline{l}^B + C_B^N \, \dot{\underline{l}}^B = \dot{C}_B^N \, \underline{l}^B + C_B^N \, \dot{\underline{l}}^B - \dot{C}_E^N \, C_B^E \, \underline{l}^B \\
&= \frac{d}{dt}\left(C_B^N \, \underline{l}^B \right) - \dot{C}_E^N \, C_B^E \, \underline{l}^B = \frac{d}{dt}\left(C_B^N \, \underline{l}^B \right) + \underline{\omega}_{EN}^N \times \left(C_B^N \, \underline{l}^B \right)
\end{aligned}
\tag{16}
$$

Substituting (16) in (6) with (7) then obtains:

$$
\underline{v}_{INS}^N = C_{N*}^N \, \underline{v}_{Ref}^{N*} + \frac{d}{dt}\left(C_B^N \, \underline{l}^B \right) + \underline{\omega}_{EN}^N \times \left(C_B^N \, \underline{l}^B \right) \tag{17}
$$

Kalman alignment filter design using an integrated velocity matching measurement is based on (17).

Lever Arm Modeling

For $\underline{v}_{INS}^N$ equations (15) and (17), the $\underline{l}^B$ lever arm term can be represented as

$$
\underline{l}^B = \underline{l}_0^B + \underline{l}_{Flex}^B \tag{18}
$$

where

$\underline{l}_0^B$ = Value of $\underline{l}^B$ at the start of Kalman alignment.

$\underline{l}_{Flex}^B$ = Small variation in $\underline{l}^B$ since the start of Kalman alignment due to structural bending (flexure) between the INS and reference navigation device.

The derivative of (18) shows that for $\dot{\underline{l}}^B$ in (15),

$$
\dot{\underline{l}}^B = \dot{\underline{l}}_{Flex}^B \tag{19}
$$

INERTIAL NAVIGATION OPERATIONS DURING KALMAN ALIGNMENT

Kalman alignment operations in the INS computer are configured as a Kalman filter aided inertial navigation operation in which ξ_{INS} inertial navigation updating operations in (1) between Kalman n cycles are from [6 - Eqs. (28) - (34)] and [2 - Sects. 8.1.1.1 - 8.1.1.2]:

$$\widehat{C}_{N*}^{N} = I + \widehat{sin\beta}\left(\underline{u}_{ZN}^{N}\times\right) + \left(1 - \widehat{cos\beta}\right)\left(\underline{u}_{ZN}^{N}\times\right)\left(\underline{u}_{ZN}^{N}\times\right)$$

$$\frac{d}{dt}\,\widehat{sin\beta} = 0 \qquad \frac{d}{dt}\,\widehat{cos\beta} = 0$$

$$\hat{\underline{\omega}}_{IE}^{N*} = \hat{\underline{\omega}}_{IE\,Ref}^{N*} \qquad \hat{\underline{\omega}}_{EN}^{N*} = \hat{\underline{\omega}}_{EN*}^{N*} = \hat{\underline{\omega}}_{EN*Ref}^{N*}$$

$$\hat{\underline{\omega}}_{IE}^{N} = \widehat{C}_{N*}^{N}\,\hat{\underline{\omega}}_{IE}^{N*} \qquad \hat{\underline{\omega}}_{EN}^{N} = \widehat{C}_{N*}^{N}\,\hat{\underline{\omega}}_{EN}^{N*}$$

$$\hat{\underline{\omega}}_{IN}^{N} = \hat{\underline{\omega}}_{IE}^{N} + \hat{\underline{\omega}}_{EN}^{N} \qquad \hat{\underline{\omega}}_{IB}^{B} = \tilde{\underline{\omega}}_{IB}^{B} - \widehat{K}_{Scal/Mis}\,\tilde{\underline{\omega}}_{IB}^{B} - \widehat{\underline{K}}_{Bias}$$

$$\dot{\widehat{C}}_{B}^{N} = \widehat{C}_{B}^{N}\left(\hat{\underline{\omega}}_{IB}^{B}\times\right) - \left(\hat{\underline{\omega}}_{IN}^{N}\times\right)\widehat{C}_{B}^{N}$$

$$\widehat{C}_{B\,n}^{N} = \widehat{C}_{B\,n-1}^{N} + \int_{t_{n-1}}^{t_n} \dot{\widehat{C}}_{B}^{N}\,dt \tag{20}$$

$$\hat{v}_{ZN_n} = \hat{v}_{ZN*Ref/n} + \underline{u}_{ZN}^{N}\cdot\left[\widehat{C}_{B_n}^{N}\left(\hat{\underline{\omega}}_{IB_n}^{B}\times\hat{\underline{l}}_{0_n}^{B}\right)\right]$$

$$\hat{\underline{g}}_{P}^{N} = \widehat{C}_{N*}^{N}\,\hat{\underline{g}}_{P\,Ref}^{N*} \qquad \hat{\underline{a}}_{SF}^{B} = \tilde{\underline{a}}_{SF}^{B} - \widehat{L}_{Scal/Mis}\,\tilde{\underline{a}}_{SF}^{B} - \widehat{\underline{L}}_{Bias}$$

$$\dot{\hat{\underline{v}}}_{INS\,H}^{N} = \left[\widehat{C}_{B}^{N}\,\hat{\underline{a}}_{SF}^{B} + \hat{\underline{g}}_{P}^{N} - \left(\hat{\underline{\omega}}_{EN}^{N} + 2\,\hat{\underline{\omega}}_{IE}^{N}\right)\times\left(\hat{\underline{v}}_{INS\,H}^{N} + \hat{v}_{ZN}\,\underline{u}_{ZN}^{N}\right)\right]_{H}$$

$$\hat{\underline{v}}_{INS\,H/n}^{N} = \hat{\underline{v}}_{INS\,H/n-1}^{N} + \int_{t_{n-1}}^{t_n} \dot{\hat{\underline{v}}}_{INS\,H}^{N}\,dt$$

where

$\widehat{(\;)}$ = Designation for a computed or measured parameter containing error compared to the same but idealized error-free ( ) parameter.

$\widetilde{(\;)}$ = Designation for inertial sensor (gyro or accelerometer) output parameter containing error compared to the sensor input error-free parameter (i.e., gyro sensed angular rate $\underline{\omega}_{IB}^{B}$ or accelerometer sensed specific force acceleration $\underline{a}_{SF}^{B}$).

β = Constant heading angle between the N and N^* frames measured positive around the upward defined N and N^* frame Z axes.

I = Identity matrix.

$\underline{\hat{\omega}}_{IE_{Ref}}^{N*}$, $\underline{\hat{\omega}}_{EN*_{Ref}}^{N*}$, $\underline{\hat{\omega}}_{EN_{Ref}}^{N*}$ = Angular rates $\underline{\omega}_{IE}$, $\underline{\omega}_{EN}$ calculated in $N*$ coordinates using standard INS computation techniques, e.g., [2 - Sects. 4.1.1 & 5.3], but based on $N*$ frame navigation data provided to the Kalman alignment process by the reference navigation device (and that N and N* rotate at the same angular rate).

$\widehat{K}_{Scal/Mis}$, $\widehat{L}_{Scal/Mis}$ = Gyro, accelerometer scale-factor/misalignment correction matrices.

$\underline{\widehat{K}}_{Bias}$, $\underline{\widehat{L}}_{Bias}$ = Gyro, accelerometer bias correction vectors.

v_{ZN} = Component of $\underline{v}^N$ along the N frame vertical Z axis.

$\hat{v}_{ZN*_{Ref}}$ = Component of $\underline{\hat{v}}_{Ref}^{N*}$ along the $N*$ frame vertical Z axis.

$\underline{a}_{SF}$ = Acceleration relative to gravitational inertial space [4], measured by strapdown INS accelerometers as $\underline{a}_{SF}^B$, generating the accelerometer output vector $\underline{\tilde{a}}_{SF}^B$ containing errors.

$\underline{\hat{g}}_{P_{Ref}}^{N*}$ = Plumb-bob gravity calculated in $N*$ coordinates using standard INS computation techniques, e.g., [2 - Sects. 5.4 & 5.4.1], but based on $N*$ frame navigation data provided to the Kalman alignment process by the reference navigation device.

H = Subscript indicating horizontal components of the designated vector.

Note in (20) that $\widehat{C}_{N*}^{N}$ is represented by the two scalar sine and cosine parameters $\widehat{sin\beta}$ and $\widehat{cos\beta}$. Errors in these parameters are part of the $\underline{\hat{x}}$ error state vector estimated by the Kalman alignment filter in (1), and are corrected (updated) at each n cycle by the control vector $\underline{\hat{u}}_{c_n}$. At Kalman alignment completion, the final estimate for $\widehat{C}_{N*}^{N}$ determines the heading of the N frame relative the known reference device $N*$ heading, and uses it to initialize heading for subsequent navigation mode operations [6].

Note that the INS vertical velocity component $\hat{v}_{ZN_n}$ in (20) is equated to vertical reference velocity $\hat{v}_{ZN*_{Ref/n}}$ as in [6], but with a correction for the estimated lever arm $\underline{\hat{l}}_{0_n}^B$ between the reference device and INS locations (based on an approximate form of (15) with (18)). The error in $\underline{\hat{l}}_{0_n}^B$ is updated in (1) by the $\underline{\hat{u}}_{c_n}$ control vector based on estimates for $\delta\underline{\hat{l}}_0^B$ error included in the $\underline{\hat{x}}$ error state vector. Note also that the $\underline{l}_{Flex}^B$ term in (18) is not present in the (20) $\hat{v}_{ZN_n}$

equation. This is because of its potential for rapid change (compared to the Kalman update cycle time), and because of the difficulty in representing $\underline{l}^B_{Flex}$ with an accurate analytical dynamic model.

INERTIAL NAVIGATION ERROR STATES DURING KALMAN ALIGNMENT

The $\hat{\underline{x}}$ error state vector components estimated in (1) during Kalman alignment consist of inertial navigation (20) parameter errors as in [6], plus inaccuracy in lever arm estimate $\hat{\underline{l}}^B_0$:

Kalman INS Alignment Estimated Error States

$$\delta K_{Scal/Mis}, \delta \underline{K}_{Bias}, \delta L_{Scal/Mis}, \delta \underline{L}_{Bias}, \Delta sin\beta, \Delta cos\beta, \underline{\gamma}^N, \Delta \underline{v}^N_{INS_H}, \delta \underline{l}^B_0 \tag{21}$$

where

$\delta(\ ) =$ Designation for errors that are small compared with $(\ )$.

$\Delta(\ ) =$ Designation for errors that can be as large as $(\ )$.

$\underline{\gamma}^N =$ Small rotation angle error vector associated with the $\hat{C}^N_B$ matrix (considering the N frame to be misaligned), as projected on N frame axes.

The $\Delta(\ )$ designation is assigned to $\Delta\sin\beta$ and $\Delta\cos\beta$ in (21) because of the initial wide angle heading uncertainty. The $\Delta(\ )$ designation for $\Delta \underline{v}^N_{INS_H}$ in (21) is used because the horizontal components of $\hat{\underline{v}}^N_{INS}$ are initialized with $\underline{v}^{N*}_{Ref}$, i.e., assuming no heading misalignment between the N and $N*$ frames [6]. The $\underline{\gamma}^N$ error is small because the N frame in $\hat{C}^N_B$ is defined to have zero heading error at the start of Kalman alignment, and because the initial leveling error in $\hat{C}^N_B$ is small by virtue of the Coarse Leveling process.

The error state dynamic equations for the (21) error states are derived in [6] (exclusive of $\underline{l}^B_0$ lever arm effects) and summarized by [6 - Eqs. (37) - (44)]. Appendix C lists the [6] estimated error state dynamic rate equations, including lever arm error $\delta \underline{l}^B_0$ in the (20) $\hat{v}_{ZN}$ equation, and equating $\widehat{\delta \dot{\underline{l}}^B_0}$ to zero (based on the definition of $\underline{l}^B_0$ being the lever arm value at the start of alignment, hence, constant as well as its error). The Appendix C equations comprise the elements of $\dot{\hat{\underline{x}}}$ in (1) for estimated error state vector propagation between Kalman update cycles.

Because of the difficulty in accurately modeling input reference data error, Appendix C and (21) as in [6] contain no allowances for reference data error estimation. However, statistically estimated input data error uncertainties can still be provisioned in the Kalman estimation process, not as errors to be estimated, but as part of the K_n Kalman gain calculations in (2) (to be discussed subsequently).

VELOCITY MATCHING OBSERVATION AND ESTIMATED MEASUREMENT VECTORS

The observation vector $\widehat{\underline{M}}_n$ in (1) for a velocity matching Kalman alignment filter is based on [6]; by comparing the horizontal components of $\hat{\underline{v}}_{INS}^{N}$ with their equivalent as formulated from (15) with (18) for $\underline{l}^B$:

$$
\begin{aligned}
\widehat{\underline{M}}_n &= \hat{\underline{v}}_{INS\,H/n}^{N} - \widehat{C}_{N*}^{N}\,\hat{\underline{v}}_{Ref\;H/n}^{N*} - \left(\widehat{C}_B^N\right)_{H/n}\left\{\left[\hat{\underline{\omega}}_{IB}^{B} - \left(\widehat{C}_B^N\right)^T \hat{\underline{\omega}}_{IE}^{N}\right]\times \hat{\underline{l}}_0^B\right\}_n \\
&\approx \hat{\underline{v}}_{INS\,H/n}^{N} - \widehat{C}_{N*}^{N}\,\hat{\underline{v}}_{Ref\;H/n}^{N*} - \left(\widehat{C}_B^N\right)_{H/n}\hat{\underline{\omega}}_{IB_n}^{B}\times \hat{\underline{l}}_{0_n}^B
\end{aligned}
\qquad (22)
$$

where

n = Subscript designation identifying a parameter value a Kalman update cycle n.

An estimate for the $\dot{\underline{l}}^B$ term in (15) is not included in (22) because from (19), it is generated by flexure rate, a parameter of notorious high frequency and random nature (compared to the Kalman alignment update frequency), with elusive representation by an accurate analytic model.

The estimated measurement vector $\hat{\underline{z}}_n$ in (1) for a velocity matching Kalman filter is derived from the equivalent error-free form of (22) obtained as $\underline{v}_{INS\,H/n}^{N}$ minus the equivalent from (15), with (18) for $\underline{l}^B$ and (19) for $\dot{\underline{l}}^B$:

$$
\begin{aligned}
\underline{M}_n = 0 &= \underline{v}_{INS\,H/n}^{N} - C_{N*}^{N}\,\underline{v}_{Ref\;H/n}^{N*} - \left(C_B^N\right)_{H/n}\left\{\left[\underline{\omega}_{IB}^{B} - \left(C_B^N\right)^T \underline{\omega}_{IE}^{N}\right]\times\left(\underline{l}_0^B + \underline{l}_{Flex}^B\right) + \dot{\underline{l}}_{Flex}^B\right\}_n \\
&\approx \underline{v}_{INS\,H/n}^{N} - C_{N*}^{N}\,\underline{v}_{Ref\;H/n}^{N*} - \left(C_B^N\right)_{H/n}\left[\underline{\omega}_{IB}^{B}\times\left(\underline{l}_0^B + \underline{l}_{Flex}^B\right) + \dot{\underline{l}}_{Flex}^B\right]_n
\end{aligned}
\qquad (23)
$$

The (23) error-free parameters can be defined as the (22) parameters (containing errors), minus the parameter errors:

$$
\begin{aligned}
&\underline{v}_{INS\,H}^{N} = \hat{\underline{v}}_{INS\,H}^{N} - \Delta\underline{v}_{INS\,H}^{N} \qquad \underline{v}_{Ref\;H}^{N*} = \hat{\underline{v}}_{Ref\;H}^{N*} - \delta\underline{v}_{Ref\;H}^{N*} \\
&C_B^N = \widehat{C}_B^N - \delta C_B^N \qquad \underline{\omega}_{IB}^{B} = \hat{\underline{\omega}}_{IB}^{B} - \delta\underline{\omega}_{IB}^{B} \qquad \underline{l}_0^B = \hat{\underline{l}}_0^B - \delta\underline{l}_0^B \\
&C_{N*}^{N} = \widehat{C}_{N*}^{N} - \Delta C_{N*}^{N}
\end{aligned}
\qquad (24)
$$

with, based on $\widehat{C}_{N*}^{N}$ in (20),

$$
\begin{aligned}
C_{N*}^{N} &= I + \sin\beta\left(\underline{u}_{ZN}^{N}\times\right) + (1-\cos\beta)\left(\underline{u}_{ZN}^{N}\times\right)\left(\underline{u}_{ZN}^{N}\times\right) \\
\Delta\sin\beta &\equiv \widehat{\sin\beta} - \sin\beta \qquad \Delta\cos\beta \equiv \widehat{\cos\beta} - \cos\beta \\
\Delta C_{N*}^{N} &= \widehat{C}_{N*}^{N} - C_{N*}^{N} = \Delta\sin\beta\left(\underline{u}_{ZN}^{N}\times\right) - \Delta\cos\beta\left(\underline{u}_{ZN}^{N}\times\right)\left(\underline{u}_{ZN}^{N}\times\right)
\end{aligned}
\tag{25}
$$

Substituting (24) into (23) and subtracting the result from (22) then finds:

$$
\begin{aligned}
\widehat{\underline{M}}_{n} &= \Delta\underline{v}_{INS\,H/n}^{N} - \Delta C_{N*}^{N}\ \hat{\underline{v}}_{Ref\ H/n}^{N*} - \left(\delta C_{B}^{N}\right)_{H/n}\left(\hat{\underline{\omega}}_{IB_n}^{B}\times\hat{\underline{l}}_{0_n}^{B}\right) \\
&- \left(\widehat{C}_{B}^{N}\right)_{H/n}\left(\delta\underline{\omega}_{IB_n}^{B}\times\hat{\underline{l}}_{0_n}^{B}\right) - \left(\widehat{C}_{B}^{N}\right)_{H/n}\left(\hat{\underline{\omega}}_{IB_n}^{B}\times\delta\underline{l}_{0_n}^{B}\right) + \left(C_{B}^{N}\right)_{H/n}\left(\underline{\omega}_{IB_n}^{B}\times\underline{l}_{Flex_n}^{B}\right) \\
&+ \left(C_{B}^{N}\right)_{H/n}\dot{\underline{l}}_{Flex_n}^{B} - \widehat{C}_{N*}^{N}\,\delta\underline{v}_{Ref\ H/n}^{N*} + \Delta C_{N*}^{N}\,\delta\underline{v}_{Ref\ H/n}^{N*} + \text{other higher order terms}
\end{aligned}
\tag{26}
$$

Reference [6 - Appendix A] shows that the δC_{B}^{N} error in (25) can be defined in terms of a small angle error vector which, to first order, approximates as

$$
\delta C_{B}^{N} \approx -\left(\underline{\gamma}^{N}\times\right)\widehat{C}_{B}^{N}
\tag{27}
$$

Substituting (27) into (26) obtains:

$$
\begin{aligned}
\widehat{\underline{M}}_{n} &= \Delta\underline{v}_{INS\,H/n}^{N} - \Delta C_{N*}^{N}\ \hat{\underline{v}}_{Ref\ H/n}^{N*} + \left(\underline{\gamma}_{Lin}^{N}\times\right)_{H/n}\widehat{C}_{B_n}^{N}\left(\hat{\underline{\omega}}_{IB_n}^{B}\times\hat{\underline{l}}_{0_n}^{B}\right) \\
&- \left(\widehat{C}_{B}^{N}\right)_{H/n}\left(\delta\underline{\omega}_{IB_n}^{B}\times\hat{\underline{l}}_{0_n}^{B}\right) - \left(\widehat{C}_{B}^{N}\right)_{H/n}\left(\hat{\underline{\omega}}_{IB_n}^{B}\times\delta\underline{l}_{0_n}^{B}\right) + \left(C_{B}^{N}\right)_{H/n}\left(\underline{\omega}_{IB_n}^{B}\times\underline{l}_{Flex_n}^{B}\right) \\
&+ \left(C_{B}^{N}\right)_{H/n}\dot{\underline{l}}_{Flex_n}^{B} - \widehat{C}_{N*}^{N}\,\delta\underline{v}_{Ref\ H/n}^{N*} + \Delta C_{N*}^{N}\,\delta\underline{v}_{Ref\ H/n}^{N*} + \text{other higher order terms}
\end{aligned}
\tag{28}
$$

Equation (28) for $\widehat{\underline{M}}_{n}$ is the identical equivalent to (22), but expressed as a function of system errors. A linearized version of $\widehat{\underline{M}}_{n}$ is obtained from (28) by deleting higher order terms, and using linearized error rates $\Delta\dot{\underline{v}}_{INS\,H}^{N}$, $\dot{\underline{\gamma}}^{N}$ for integration into $\Delta\underline{v}_{INS\,H/n}^{N}$, $\underline{\gamma}^{N}$:

$$
\begin{aligned}
\underline{z}_{n} &= \Delta\underline{v}_{INS\,Lin/H/n}^{N} - \Delta C_{N*}^{N}\ \hat{\underline{v}}_{Ref\ H/n}^{N*} \\
&+ \left(\underline{\gamma}_{Lin}^{N}\times\right)_{H/n}\widehat{C}_{B_n}^{N}\left(\hat{\underline{\omega}}_{IB_n}^{B}\times\hat{\underline{l}}_{0_n}^{B}\right) - \left(\widehat{C}_{B}^{N}\right)_{H/n}\left(\delta\underline{\omega}_{IB_n}^{B}\times\hat{\underline{l}}_{0_n}^{B}\right) - \left(\widehat{C}_{B}^{N}\right)_{H/n}\left(\hat{\underline{\omega}}_{IB_n}^{B}\times\delta\underline{l}_{0_n}^{B}\right) \\
&+ \left(\widehat{C}_{B}^{N}\right)_{H/n}\left(\hat{\underline{\omega}}_{IB_n}^{B}\times\underline{l}_{Flex_n}^{B}\right) + \left(\widehat{C}_{B}^{N}\right)_{H/n}\dot{\underline{l}}_{Flex_n}^{B} - \widehat{C}_{N*}^{N}\,\delta\underline{v}_{Ref\ H/n}^{N*}
\end{aligned}
\tag{29}
$$

where

Lin = Subscript designating an error parameter obtained by integrating a corresponding linearized error dynamic rate equation provided in Appendix C

$\underline{z}_n$ = Linearized version of $\underline{\widehat{M}}_n$.

Note that the $\Delta C_{N*}^{N}\, \delta \underline{v}_{Ref\,H/n}^{N*}$ second order term in (28) has been dropped in (29), even though its magnitude is initially first order by virtue of the large value of ΔC_{N*}^{N} at the start of alignment. The rationale is that $\delta \underline{v}_{Ref\,H/n}^{N*}$ is generally very small, hence, neglecting $\Delta C_{N*}^{N}\, \delta \underline{v}_{Ref\,H/n}^{N*}$ will have negligible impact on early Kalman alignment convergence when ΔC_{N*}^{N} is large and has high visibility on the measurement. As Kalman alignment convergence proceeds and the small $\delta \underline{v}_{Ref\,H/n}^{N*}$ error becomes important for estimating small system error effects, the ΔC_{N*}^{N} estimation error will have converged to a small value, making $\Delta C_{N*}^{N}\, \delta \underline{v}_{Ref\,H/n}^{N*}$ second order and safely neglected. A more sophisticated treatment of neglecting $\Delta C_{N*}^{N}\, \delta \underline{v}_{Ref\,H/n}^{N*}$ in (29) accounts for its presence as adaptive second order measurement noise [3].

The estimated measurement vector $\hat{\underline{z}}_n$ used for measurement residual $\underline{z}_{Res_n}$ determination in (1) (by comparison with $\underline{\widehat{M}}_n$), is (29) with estimates for the error parameters, and neglecting the flexure terms because they are random, hence, unknown:

$$\begin{aligned}\hat{\underline{z}}_n = \widehat{\Delta \underline{v}_{INS\,Lin/H/n}^{N}} - \widehat{\Delta C_{N*}^{N}}\ \hat{\underline{v}}_{Ref\,H/n}^{N*} + \left(\widehat{\underline{\gamma}_{Lin}^{N}} \times\right)_{H/n} \hat{C}_{B_n}^{N} \left(\hat{\underline{\omega}}_{IB_n}^{B} \times \hat{\underline{l}}_{0_n}^{B}\right) \\ - \left(\hat{C}_{B}^{N}\right)_{H/n} \left(\widehat{\delta \underline{\omega}_{IB_n}^{B}} \times \hat{\underline{l}}_{0_n}^{B}\right) - \left(\hat{C}_{B}^{N}\right)_{H/n} \left(\hat{\underline{\omega}}_{IB_n}^{B} \times \widehat{\delta \underline{l}_{0\,n}^{B}}\right)\end{aligned} \tag{30}$$

The measurement matrix $\widehat{H}_n$ for calculating $\hat{\underline{z}}_n$ from $\hat{\underline{x}}_n$ in (1) would be formed from the error coefficients in (30). The $\hat{\underline{x}}_n$ elements for (30) are calculated as part of the (1) integration/estimation/control process using Appendix C for $\dot{\hat{\underline{x}}}$ error state dynamic rates in (1).

Note that as in (21), that velocity reference error $\delta \underline{v}_{Ref\,H}^{N*}$ has not been included in (30) for Kalman alignment estimation. For velocity reference data provided within a "tightly coupled" GPS/INS Kalman aiding configuration, the GPS receiver input to the INS computer would be pseudo-range measurements to GPS satellites, containing GPS receiver clock phase/frequency error. The GPS/INS Kalman filter would be structured to include estimation/correction of the

GPS clock error during INS alignment. Thus, an estimatable equivalent to $\delta \underline{v}^{N*}_{Ref\,H}$ would in principle, be included in the GPS/INS tightly-coupled Kalman measurement model as a function of estimated GPS clock phase/frequency error. As of this writing, however, a GPS/INS tightly-coupled Kalman alignment filter has yet to formulated from [6], for wide initial heading angle error application.

For a traditional “loosely coupled” GPS/INS configuration, GPS clock frequency/phase error is corrected within the GPS receiver system, and $\underline{v}^{N*}_{Ref\,H}$ would be the output provided to the INS. However, for lack of an accurate GPS receiver processing model, estimating $\delta \underline{v}^{N*}_{Ref\,H}$ in this case is usually not possible, and would typically be neglected as in (30). A random presence of $\delta \underline{v}^{N*}_{Ref\,H}$ in the measurement can still be accounted for, however, as part of Kalman gain calculation in (2) (to be discussed subsequently).

INTEGRATED VELOCITY MATCHING OBSERVATION AND ESTIMATED MEASUREMENT VECTORS

The observation and estimated measurement vectors ($\widehat{\underline{M}}_n$ and $\hat{\underline{z}}_n$) in (1) for an integrated velocity matching Kalman alignment approach are formulated from the integrated difference between the INS horizontal velocity and its equivalent using (17) with (18) for $\underline{l}^B$. For the idealized error-free form of the observation vector, the integrated velocity comparison $\underline{M}_n$ is:

$$\begin{aligned}\underline{M}_n = 0 &= \int_0^{t_n} \left\{ \underline{v}^N_{INS} - C^N_{N*}\, \underline{v}^{N*}_{Ref} - \frac{d}{dt}\left[C^N_B \left(\underline{l}^B_0 + \underline{l}^B_{Flex} \right) \right] - \underline{\omega}^N_{EN} \times \left[C^N_B \left(\underline{l}^B_0 + \underline{l}^B_{Flex} \right) \right] \right\}_H dt \\ &= \int_0^{t_n} \left\{ \underline{v}^N_{INS} - C^N_{N*}\, \underline{v}^{N*}_{Ref} - \underline{\omega}^N_{EN} \times \left[C^N_B \left(\underline{l}^B_0 + \underline{l}^B_{Flex} \right) \right] \right\}_H dt \\ &\qquad - C^N_{B_{H/n}} \left(\underline{l}^B_0 + \underline{l}^B_{Flex_n} \right) + C^N_{B_{H/0}}\, \underline{l}^B_0 \end{aligned} \tag{31}$$

in which it has been recognized from the definition of $\underline{l}^B_0$, that $\underline{l}^B_{Flex}$ is zero at the start of alignment, and where

t = Time from the start of Kalman alignment.

Equation (31) simplifies by defining

$$\underline{S} \equiv \int_0^t \dot{\underline{S}}\, dt \qquad \begin{aligned} \dot{\underline{S}} = 0 &= \left\{ \underline{v}^N_{INS} - C^N_{N*}\, \underline{v}^{N*}_{Ref} - \underline{\omega}^N_{EN} \times \left[C^N_B \left(\underline{l}^B_0 + \underline{l}^B_{Flex} \right) \right] \right\}_H \\ &\approx \left[\underline{v}^N_{INS} - C^N_{N*}\, \underline{v}^{N*}_{Ref} - \underline{\omega}^N_{EN} \times \left(C^N_B\, \underline{l}^B_0 \right) \right]_H \end{aligned} \tag{32}$$

so that (31) becomes

$$\underline{M}_n = 0 = \underline{S}_n - \left(C^N_{BH/n} - C^N_{BH/0} \right) \underline{l}^B_0 - C^N_{BH/n} \, \underline{l}^B_{Flex_n} \tag{33}$$

Recognizing from the definition of the N and N^* frames that $C^N_{N^*}$ is constant, an equivalent recursive form of $\underline{S}_n$ can be structured for (33) based on (32) as:

$$\underline{S}_n = 0 = \underline{S}_{n-1} + \int_{t_{n-1}}^{t_n} \left[\underline{v}^N_{INS\,H} - \left(\underline{\omega}^N_{EN} \times \right)_H C^N_B \, \underline{l}^B_0 \right] dt - C^N_{N^*} \int_{t_{n-1}}^{t_n} \underline{v}^{N^*}_{Ref\,H} dt \tag{34}$$

The (34) form is preferred as the basis for deriving an equation for the observation vector in (1) because, as will be discussed in the next section, it allows the option for an accurate evaluation of the $\int_{t_{n-1}}^{t_n} \underline{v}^{N^*}_{Ref} \, dt$ term using n cycle positioning data provided by the reference navigation device.

Observation Vector

The INS calculated observation vector $\widehat{\underline{M}}_n$ for (1) is obtained from (33) and (34) by substituting actual values (containing errors) for the idealized error-free parameters, deleting the $\underline{l}^B_{Flex}$ term in $\underline{M}_n$ as non-estimatable (because of its random nature with high frequency components compared to the Kalman filter updating frequency), and deleting the zero equalities in (33) and (34) because, unlike the idealized version, the actual observation contains errors, hence, is generally non-zero:

$$\widehat{\underline{S}}_n = \widehat{\underline{S}}_{n-1} + \int_{t_{n-1}}^{t_n} \left[\hat{\underline{v}}^N_{INS\,H} - \left(\hat{\underline{\omega}}^N_{EN} \times \right)_H \widehat{C}^N_B \, \hat{\underline{l}}^B_0 \right] dt - \widehat{C}^N_{N^*_n} \int_{t_{n-1}}^{t_n} \hat{\underline{v}}^{N^*}_{Ref\,H} \, dt \tag{35}$$

$$\widehat{\underline{M}}_n = \widehat{\underline{S}}_n - \left(\widehat{C}^N_{BH/n} - \widehat{C}^N_{BH/0} \right) \hat{\underline{l}}^B_{0_n} \tag{36}$$

Equation (35) for $\widehat{\underline{S}}_n$ allows an accurate approximation for the $\int_{t_{n-1}}^{t_n} \hat{\underline{v}}^{N^*}_{Ref\,H} \, dt$ term in applications when $\hat{\underline{v}}^{N^*}_{Ref\,H}$ is not available at a high enough rate to effect an accurate digital integration process (i.e., when $\hat{\underline{v}}^{N^*}_{Ref\,H}$ contains high frequency components compared to the Kalman alignment filter update frequency). Under such conditions, the positioning output from the reference device can be used to derive $\hat{\underline{v}}^{N^*}_{Ref\,H}$ based on the development in Appendix B:

$$
\begin{aligned}
\int_{t_{n-1}}^{t_n} \underline{v}_{Ref\,H}^{N*}\, dt &= \left[\left(\hat{C}_E^{N*} \right)_n \left(\underline{\hat{R}}_{Ref\,n}^{E} - \underline{\hat{R}}_{Ref\,n\text{-}1}^{E} \right) + \int_{t_{n-1}}^{t_n} \left(\underline{\hat{\omega}}_{EN}^{N*} \times \right) \hat{C}_E^{N*} \left(\underline{\hat{R}}_{Ref}^{E} - \underline{\hat{R}}_{Ref\,n\text{-}1}^{E} \right) \right]_H dt \\
&\approx \left\langle \left\{ \left(\hat{C}_E^{N*} \right)_n + \frac{1}{4} \left[\left(\underline{\hat{\omega}}_{EN\,n}^{N*} \times \right) \left(\hat{C}_E^{N*} \right)_n + \left(\underline{\hat{\omega}}_{EN\,n\text{-}1}^{N} \times \right) \left(\hat{C}_E^{N*} \right)_{n-1} \right] T_n \right\} \left(\underline{\hat{R}}_{Ref\,n}^{E} - \underline{\hat{R}}_{Ref\,n\text{-}1}^{E} \right) \right\rangle_H
\end{aligned}
\tag{37}
$$

where

T_n = Kalman cycle time interval (from t_{n-1} to t_n).

The $\hat{C}_E^{N*}$ and $\underline{\hat{R}}_{Ref}^{E}$ terms in (37) are functions of standard position outputs provided by the navigation reference device (e.g., latitude, longitude, and altitude) and are readily calculated for $\int_{t_{n-1}}^{t_n} \underline{v}_{Ref\,H}^{N*}\, dt$ determination; e.g., the transpose of [2 - Eq. (4.4.2.1-2)] for $\hat{C}_E^{N*}$, and [2 - Eq. (5.2.2-1)] for $\underline{\hat{R}}_{Ref}^{E}$ using [2 - Eq. (4.4.2.2-4)] for $\underline{u}_{Up}^{E} = \underline{u}_{ZN}^{E}$ and [2 - Eq. (5.1-10)] for $\underline{R}_S^{'}$.

Estimated Measurement Vector

The estimated measurement vector $\hat{\underline{z}}_n$ in (1) for an integrated velocity matching Kalman alignment filter is derived from the (33) error-free observation form by first setting the (33) parameters equal to the difference between their computed values (containing errors) minus the associated errors, subtracting the result from $\underline{\widehat{M}}_n$ in (36), and noting that $\underline{M}_n$ in (33) is zero:

$$
\underline{S}_n = \hat{\underline{S}}_n - \Delta \underline{S}_n \qquad C_{BH}^{N} = \hat{C}_{BH}^{N} - \delta C_{BH}^{N} \qquad \underline{l}_{0_n}^{B} = \hat{\underline{l}}_{0_n}^{B} - \delta \underline{l}_{0_n}^{B} \tag{38}
$$

$$
\underline{\widehat{M}}_n = \Delta \underline{S}_n - \left(\delta C_{BH/n}^{N} - \delta C_{BH/0}^{N} \right) \hat{\underline{l}}_{0_n}^{B} - \left(\hat{C}_{BH/n}^{N} - \hat{C}_{BH/0}^{N} \right) \delta \underline{l}_{0_n}^{B} + C_{BH/n}^{N}\, \underline{l}_{Flex_n}^{B} + h.o.t. \tag{39}
$$

where

$h.o.t.$ = Higher order terms.

The δC_{BH}^{N} terms in (39) can be approximated [6 - Appendix A] in terms of the small angle error $\underline{\gamma}^N$ in $\hat{C}_B^N$:

$$
\delta C_{BH/n}^{N} \approx -\left(\underline{\gamma}_n^N \times \right)_H \hat{C}_{B_n}^{N} \qquad \delta C_{BH/0}^{N} \approx -\left(\underline{\gamma}_0^N \times \right)_H \hat{C}_{B0}^{N} \tag{40}
$$

Substitution in (39) then finds:

$$\underline{\widehat{M}}_n = \Delta\underline{S}_n + \left[\left(\underline{\gamma}_n^N \times\right)_H \widehat{C}_{B_n}^N - \left(\underline{\gamma}_0^N \times\right)_H \widehat{C}_{B_{H/0}}^N \right] \underline{\hat{l}}_{0_n}^B - \left(\widehat{C}_{B_{H/n}}^N - \widehat{C}_{B_{H/0}}^N\right) \delta\underline{l}_{0_n}^B + C_{B_{H/n}}^N \underline{l}_{Flex_n}^B + h.o.t. \tag{41}$$

Equation (41) is the exact equivalent to observation equation (36), but expressed as a function of parameter errors that make the computed observation non-zero (as opposed to the idealized error-free form in (33) for $\underline{M}_n$ that was constructed to be identically zero).

The "measurement vector" $\underline{z}_n$ is the linearized form of (41):

$$\underline{z}_n = \Delta\underline{S}_n + \left[\left(\underline{\gamma}_{Lin_n}^N \times\right)_H \widehat{C}_{B_n}^N - \left(\underline{\gamma}_0^N \times\right)_H \widehat{C}_{B_{H/0}}^N \right] \underline{\hat{l}}_{0_n}^B - \left(\widehat{C}_{B_{H/n}}^N - \widehat{C}_{B_{H/0}}^N\right) \delta\underline{l}_{0_n}^B + C_{B_{H/n}}^N \underline{l}_{Flex_n}^B \tag{42}$$

The estimated measurement vector $\hat{\underline{z}}_n$ in (1) is obtained from (42) using estimated values for the error parameters, with the $\underline{l}_{Flex_n}^B$ random term deleted as not estimatable:

$$\hat{\underline{z}}_n = \widehat{\Delta\underline{S}}_n + \left[\left(\widehat{\underline{\gamma}_{Lin_n}^N} \times\right)_H \widehat{C}_{B_n}^N - \left(\widehat{\underline{\gamma}_{Lin0}^N} \times\right)_H \widehat{C}_{B_{H/0}}^N \right] \underline{\hat{l}}_{0_n}^B - \left(\widehat{C}_{B_{H/n}}^N - \widehat{C}_{B_{H/0}}^N\right) \widehat{\delta\underline{l}_{0\,n}^B} \tag{43}$$

The error coefficients in (43) form the elements of measurement matrix $\underline{\widehat{H}}_n$ in (1). The estimated errors in (43) are elements of estimated error state vector $\hat{\underline{x}}$ used in (1) to calculate $\hat{\underline{z}}_n$.

Equation (43) shows that $\widehat{\Delta\underline{S}}_n$, $\widehat{\underline{\gamma}_{Lin_n}^N}$, $\widehat{\underline{\gamma}_{Lin0}^N}$, and $\widehat{\delta\underline{l}_{0\,n}^B}$ are required to form the estimated measurement, necessitating the addition of $\widehat{\Delta\underline{S}}_n$, $\widehat{\underline{\gamma}_{Lin0}^N}$ to the (21) estimated error states in $\hat{\underline{x}}$ for the velocity matching Kalman alignment filter. The corresponding additional estimated error state dynamic rate components for $\dot{\hat{\underline{x}}}$ in (1) (additions to Appendix C) derive from the estimated value of $\dot{\underline{S}}$ in (32) with $\underline{\gamma}_{Lin0}^N$ constant:

$$\dot{\hat{\underline{S}}} = \left[\hat{\underline{v}}_{INS}^N - \widehat{C}_{N*}^N \hat{\underline{v}}_{Ref}^{N*} - \hat{\underline{\omega}}_{EN}^N \times \left(\widehat{C}_B^N \underline{\hat{l}}_0^B \right) \right]_H \qquad \dot{\widehat{\underline{\gamma}_{Lin0}^N}} = 0 \tag{44}$$

Substituting from (24) and (38) for the parameters in (32), and subtracting the (32) result from (44) obtains:

$$\Delta\dot{\underline{S}} = \Delta\underline{v}^N_{INS\,H} - \Delta C^N_{N*}\,\hat{\underline{v}}^{N*}_{Ref\,H} - \left(\Delta\underline{\omega}^N_{EN}\times\right)_H \widehat{C}^N_B\,\hat{\underline{l}}^B_0 - \left(\hat{\underline{\omega}}^N_{EN}\times\right)_H \delta C^N_B\,\hat{\underline{l}}^B_0$$

$$- \left(\hat{\underline{\omega}}^N_{EN}\times\right)_H \widehat{C}^N_B\,\delta\underline{l}^B_0 - \widehat{C}^N_{N*}\,\delta\underline{v}^{N*}_{Ref\,H} + \Delta C^N_{N*}\,\delta\underline{v}^{N*}_{Ref\,H} + \text{ other higher order terms} \qquad (45)$$

$$\approx \Delta\underline{v}^N_{INS\,H} - \Delta C^N_{N*}\,\hat{\underline{v}}^{N*}_{Ref\,H} - \widehat{C}^N_{N*}\,\delta\underline{v}^{N*}_{Ref\,H} + \Delta C^N_{N*}\,\delta\underline{v}^{N*}_{Ref\,H} + \text{ other higher order terms}$$

or with linearization:

$$\Delta\dot{\underline{S}}_{Lin} = \Delta\underline{v}^N_{INS\,H/Lin} - \Delta C^N_{N*}\,\hat{\underline{v}}^{N*}_{Ref\,H} - \widehat{C}^N_{N*}\,\delta\underline{v}^{N*}_{Ref\,H} \qquad (46)$$

The comments following (29) also apply to (46) regarding eliminating $\Delta C^N_{N*}\,\delta\underline{v}^{N*}_{Ref\,H}$ in the linearization process from (45) to (46). In this case, the presence of $\Delta C^N_{N*}\,\delta\underline{v}^{N*}_{Ref\,H}$ in (46) can be represented as adaptive second order <u>process</u> noise [3], rather than adaptive <u>measurement</u> noise for (29).

The estimated version of $\Delta\dot{\underline{S}}$ for $\dot{\hat{\underline{x}}}$ in (1) is (46) using estimates for the error parameters:

$$\widehat{\Delta\dot{\underline{S}}_{Lin}} = \widehat{\Delta\underline{v}^N_{INS\,Lin/H}} - \widehat{\Delta C^N_{N*}}\,\hat{\underline{v}}^{N*}_{Ref\,H} \qquad (47)$$

In (47) as in (30), velocity reference error $\delta\underline{v}^{N*}_{Ref\,H}$ has not been included in (47) for Kalman alignment estimation. The rationale is the same as in the paragraphs following (30). As in (30), a random presence of $\delta\underline{v}^{N*}_{Ref\,H}$ in (47) can be accounted for, as part of the Kalman gain calculation in (2) (to be discussed subsequently).

The coefficients for the error terms in (47) constitute elements of the error state dynamic matrix $\widehat{A}$ in (1) associated with $\widehat{\Delta\underline{S}_{Lin}}$ propagation between Kalman cycles. From (44) for $\widehat{\dot{\underline{\gamma}}^N_{Lin0}}$, the rows of the $\widehat{A}$ matrix in (1) for $\underline{\gamma}^N_{Lin0}$ are zero.

Because dynamic changes in the $\underline{\gamma}^N_{Lin}$ error are much smaller than the initial value over the alignment period, the $\underline{\gamma}^N_{Lin0}$ error in (43) can be safely approximated by $\underline{\gamma}^N_{Lin}$, thereby eliminating the need to include $\underline{\gamma}^N_{Lin0}$ as an error state to be estimated. If this approximation is used, however, it should be recognized that $\underline{u}_c$ control vector components in (1) for $\widehat{C}^N_B$ updating should also be applied to $\widehat{C}^N_{B0}$ updating in measurement equation (43) and observation equation

(36) (as they would have if $\underline{\gamma}_{Lin0}^{N}$ was estimated separately and used for $\hat{C}_{B0}^{N}$ updating) . Using the $\underline{\gamma}_{Lin0}^{N} \approx \underline{\gamma}_{Lin}^{N}$ approach, (42) and (43) then simplify to

$$\underline{z}_n = \Delta\underline{S}_n + \left(\underline{\gamma}_{Lin_n}^{N} \times\right)_H \left(\hat{C}_{B_n}^{N} - \hat{C}_{BH/0}^{N}\right) \hat{\underline{l}}_{0_n}^{B} - \left(\hat{C}_{BH/n}^{N} - \hat{C}_{BH/0}^{N}\right) \delta\underline{l}_{0_n}^{B} + C_{BH/n}^{N} \, \underline{l}_{Flex_n}^{B} \quad (48)$$

$$\hat{\underline{z}}_n = \widehat{\Delta\underline{S}}_n + \left(\hat{\underline{\gamma}}_{Lin_n}^{N} \times\right)_H \left(\hat{C}_{B_n}^{N} - \hat{C}_{BH/0}^{N}\right) \hat{\underline{l}}_{0_n}^{B} - \left(\hat{C}_{BH/n}^{N} - \hat{C}_{BH/0}^{N}\right) \widehat{\delta\underline{l}_{0}^{B}}_{n} \quad (49)$$

MEASUREMENT NOISE

The measurement noise matrix R_M used in (2) to calculate the K_n Kalman gain matrix, represents the presence of noise on measurement residual $\underline{z}_{Res_n}$ in (1) used for estimated error state vector $\hat{\underline{x}}_n$ updating. The relationship between measurement noise and R_M is based on approximating the observation vector $\widehat{\underline{M}}_n$ in (1) by the linearized measurement vector $\underline{z}_n$, as in (29) and (42):

$$\underline{z}_n \equiv \widehat{\underline{M}}_{Lin_n} = \hat{H}_n \, \underline{x}_n + \hat{G}_{M_n} \, \underline{n}_{M_n} \quad (50)$$

where

$\underline{n}_{M_n}$ = Vector of independent random components that are uncorrelated between Kalman update cycles.

Using the $\hat{\underline{z}}_n$ equation in (1) with (50) to calculate the equivalent linearized version of $\underline{z}_{Res_n}$ in (1) finds:

$$\underline{z}_{Res\,Lin/n} = -\hat{H}_n \, \underline{\chi}_n(-) + \hat{G}_{M_n} \, \underline{n}_{M_n} \qquad \underline{\chi} \approx \hat{\underline{x}} - \underline{x} \quad (51)$$

where

$\underline{\chi}$ = Uncertainty in $\hat{\underline{x}}$.

Using $\underline{z}_{Res\,Lin/n}$ for $\underline{z}_{Res_n}$ in the (1) equation for $\hat{\underline{x}}_n(+)$, and subtracting the true value identity $\underline{x}_{n+1}(+) = \underline{x}_n(-)$ from the result, then obtains for the change in $\underline{\chi}$ from the Kalman update:

$$\underline{\chi}_n(+) = \left(I - K_n \, \hat{H}_n\right) \underline{\chi}_n(-) + K_n \, \hat{G}_{M_n} \, \underline{n}_{M_n} \quad (52)$$

The $P_n(+)$ covariance updating operation in (2) is derived from (52) by substitution into the P definition equation and expansion [2 - Sect. 15.1.2.1]:

$$P \equiv \mathcal{E}\left(\underline{\chi}\ \underline{\chi}^T\right) \qquad R_M \equiv \mathcal{E}\left(\underline{n}_M\ \underline{n}_M^T\right) \tag{53}$$

The K_n Kalman "optimal" gain equation in (2) is derived as being the K_n that minimizes $P_n(+)$ in (2) for a given $P_n(-)$ [2 - Sect. 15.1.2.1].

Velocity Matching Kalman Alignment Measurement Noise

For a velocity matching Kalman alignment filter, (50) is provided by (29). In (29), the $\widehat{G}_{M_n} \underline{n}_{M_n}$ measurement noise term in (50) would comprise $\Delta \underline{v}^N_{Lin/INS_{H/n}}$ accelerometer quantization noise (as in [6]), and $\underline{l}^B_{Flex_n}$, $\dot{\underline{l}}^B_{Flex_n}$, $\delta \underline{v}^{N*}_{Ref_{H/n}}$ in the last line of (29): $\left(\widehat{C}^N_B\right)_{H/n} \left(\widehat{\underline{\omega}}^B_{IB_n} \times \underline{l}^B_{Flex_n}\right) + \left(\widehat{C}^N_B\right)_{H/n} \dot{\underline{l}}^B_{Flex_n} - \widehat{C}^N_{N*}\, \delta \underline{v}^{N*}_{Ref_{H/n}}$. The $\widehat{G}_{M_n}$ measurement noise dynamic coupling matrix in (50) would have unity for the quantization noise rows, and would be formed from the coefficients of $\underline{n}_{M_n}$ in the previous expression for the $\underline{l}^B_{Flex_n}$, $\dot{\underline{l}}^B_{Flex_n}$, $\delta \underline{v}^{N*}_{Ref_{H/n}}$ rows. Because the definition for $\underline{n}_{M_n}$ is a column of independent random components, a difficulty arises for the $\underline{l}^B_{Flex_n}$ and $\dot{\underline{l}}^B_{Flex_n}$ terms in $\underline{n}_{M_n}$ because of their correlation (i.e., $\underline{l}^B_{Flex}$ is the integral of $\dot{\underline{l}}^B_{Flex}$). A simple expedient has been to estimate independent values for each in (52) and assume no correlation. Additionally, to assure that the flexure measurement noise model is uncorrelated from Kalman cycle to cycle (the analytical basis for $\underline{n}_{M_n}$), the Kalman cycle time should be set sufficiently higher (e.g., two times) than the lowest expected flexure mode frequency.

A more interesting problem arises for the $\delta \underline{v}^{N*}_{Ref_{H/n}}$ term because as discussed earlier, it may not be modelable as part of the estimated error states, but may be correlated between n cycles, thereby violating the defining requirement for $\underline{n}_{M_n}$. One method for dealing with a random nature for $\delta \underline{v}^{N*}_{Ref_{H/n}}$, is to assume it is correlated with itself over some time period, but that the correlation time is short compared with the Kalman update n cycle period. This would require an assumed $\delta \underline{v}^{N*}_{Ref_{H/n}}$ correlation time, with the Kalman update period then set, for example, to twice the assumed correlation time value, thus effectively un-correlating $\delta \underline{v}^{N*}_{Ref_{H/n}}$ from n cycle to n cycle. A more sophisticated method assumes a correlation model for $\delta \underline{v}^{N*}_{Ref_{H/n}}$ (e.g., a first order Markov process, e.g., [2 - Eq. (12.5.6-3)], and treating $\delta \underline{v}^{N*}_{Ref_{H/n}}$ as part of the error state vector, but not to be estimated. This can easily be achieved using a "considered variable" approach based on the (2) covariance update implementation.

With the considered variable method, the Kalman gain matrix K_n is calculated as if $\delta \underline{v}_{Ref\,H/n}^{N*}$ was part of the $\underline{x}$ error states (with a corresponding error state dynamic rate equation). Following the K_n computation, the K_n rows for $\delta \underline{v}_{Ref\,H/n}^{N*}$ are then set to zero, thus nullifying potential $\delta \underline{v}_{Ref\,H/n}^{N*}$ estimation, and eliminating the need to include $\delta \underline{v}_{Ref\,H/n}^{N*}$ in (1) as part of $\hat{\underline{x}}_n$ or $\hat{\underline{z}}_n$. The $\delta \underline{v}_{Ref\,H/n}^{N*}$ random presence then only registers as part of the covariance propagation process (third equation in (2)) and by its impact on the non-$\delta \underline{v}_{Ref\,H/n}^{N*}$ rows in K_n for $\hat{\underline{x}}_n$ updating. Further discussion of $\delta \underline{v}_{Ref\,H/n}^{N*}$ considered variable modeling is provided in the Process Noise section to follow.

The considered variable approach is only valid if the $P_n(+)$ update equation is implemented as shown in (2) (the "Joseph's" form) whose derivation is general for any K_n matrix. In some applications, a simplified form of the $P_n(+)$ equation is used that is derived from the Joseph's form by analytically substituting the optimum (2) equation for K_n (without allowance for zeroing rows) [2 - Eq. (15.1.2.1.1-4)]:

$$P_n(+) = \left(I - K_n\, H_n\right) P_n(-) \tag{54}$$

Using (54) to determine $P_n(+)$ with a K_n adjusted for zeroed rows would generate an invalid $P_n(+)$ update, producing error in subsequent calculations, and potential Kalman-loop instability. A similar problem would arise if $P_n(+)$ was calculated based on "UD factorization" [8 - Sect. 9.5], another approach used by some analysts, but also formulated assuming an optimum K_n, without allowances for zeroing rows.

<u>Integrated Velocity Matching Kalman Alignment Measurement Noise</u>

For an integrated velocity matching Kalman alignment approach, (50) is provided by (42), with $C_{B\,H/n}^{N}\, \underline{l}_{Flex_n}^{B}$ in (42) representing the $\widehat{G}_{M\,n}\, \underline{n}_{M\,n}$ measurement noise term in (50). Then the $\widehat{G}_{M\,n}$ measurement noise dynamic coupling matrix would be $C_{B\,H/n}^{N}$ and $\underline{l}_{Flex_n}^{B}$ would be $\underline{n}_{M\,n}$. The components of $\underline{l}_{Flex_n}^{B}$ would be modeled as independent random variables that are uncorrelated from n cycle to n cycle. To assure $\underline{n}_{M\,n}$ model validity, the Kalman cycle time should be set higher (e.g., two times) than the lowest expected flexure mode frequency.

PROCESS NOISE

The process noise matrix Q_P used in (2) for P covariance propagation between Kalman updates, represents the effect of noise integrated into parameters forming the observation vector $\underline{\widehat{M}}_n$ in (1). The relationship between the integrated noise and Q_P is based on approximating $\underline{\widehat{M}}_n$ by the linearized measurement vector $\underline{z}_n$ in (50). Integrated process noise is generated in $\underline{z}_n$ from its buildup in $\underline{x}_n$ between Kalman cycles according to the equivalent of $\dot{\hat{\underline{x}}}$ estimation equation in (1) that includes non-estimatable random components:

$$\dot{\underline{x}} = \widehat{A}\,\underline{x} + \widehat{G}_P\,\underline{n}_P \tag{55}$$

where

$\underline{n}_P$ = Vector of independent white noise components.

$\widehat{G}_P$ = Estimated process noise dynamic coupling matrix.

The difference between (55) and $\dot{\hat{\underline{x}}}$ provides an equation for the build-up rate in $\underline{\mathrm{x}}$ uncertainty between Kalman cycles:

$$\dot{\underline{\chi}} = \widehat{A}\,\underline{\chi} - \widehat{G}_P\,\underline{n}_P \tag{56}$$

The covariance rate $\dot{P}$ in (2) is obtained by substituting (56) into the derivative (53) for *P*, and expanding:

$$\dot{P} = \mathcal{E}\left(\dot{\underline{\chi}}\,\underline{\chi}^T + \underline{\chi}\,\dot{\underline{\chi}}^T\right) = \widehat{A}\;P + P\,\widehat{A}^T + \widehat{G}_P\,Q_P\,\widehat{G}_P^T \tag{57}$$

where

Q_P = Diagonal matrix with diagonal elements equal to the $\underline{n}_P$ white noise densities.

For the Kalman alignment filter, integrated process noise is created by random components in the gyro/accelerometer $\underline{\tilde{a}}_{SF}^B$, $\underline{\tilde{\omega}}_{IB}^B$ output vectors in (20), generating error build-up in calculated velocity $\hat{\underline{v}}_{INS\,H}^N$ and attitude $\widehat{C}_{B\,n}^N$, and ultimately entering the $\underline{\widehat{M}}_n$ observation vector input to the Kalman filter. Reference [6] shows how the $\widehat{G}_P$ and Q_P matrices in (2) are formed from gyro/accelerometer process noise elements in a velocity matching Kalman alignment application. For an integrated velocity matching approach, the [6] gyro/accelerometer process noise results also apply, but with additional random components on $\Delta\underline{v}_{INS\,H}^N$ and $\widehat{C}_{N*}^N\,\delta\underline{v}_{Ref}^{N*}$ in the (46) $\Delta \dot{\underline{S}}$ expression. (Recall that the integral of $\Delta \dot{\underline{S}}$ in (45) generates $\Delta \underline{S}_n$ in the (41) form of observation $\underline{\widehat{M}}_n$.)

Accelerometer quantization $\delta \underline{\upsilon}_{Quant}$ is a random element in $\Delta \underline{v}^N_{INS\,H}$ that directly impacts $\Delta \dot{\underline{S}}$ as white process noise. It is modeled within $\Delta \underline{v}^N_{INS\,H}$ as $\widehat{C}^N_{BH}\, \delta \underline{\upsilon}_{Quant}$ [2 - Eq. (12.5-13)]. (In [6] and discussed in the previous section, the same term is treated as measurement noise for velocity matching Kalman alignment.) For the $\Delta \dot{\underline{S}}$ portion of $\dot{\underline{x}}$ in (55) and the $\Delta \dot{\underline{S}}$ covariance equivalent of $\dot{P}$ in (57) and (2), the elements in $\widehat{G}_P$ for $\delta \underline{\upsilon}_{Quant}$ quantization noise coupling into $\dot{\underline{x}}$ (and $\dot{P}$) would be $\widehat{C}^N_{BH}$. Elements would then be included in Q_P for (57) and (2) equal to the $\delta \underline{\upsilon}_{Quant}$ quantization noise densities, a function of the digital integration algorithm update frequency for implementing $\int_{t_{n-1}}^{t_n} \left[\hat{\underline{v}}^N_{INS\,H} - \left(\hat{\underline{\omega}}^N_{EN} \times \right)_H \widehat{C}^N_B\, \hat{\underline{l}}^B_0 \right] dt$ in $\hat{\underline{S}}_n$ equation (35) - See [2 - Sect. 16.2.3.1] for the equivalent effect when calculating position by velocity integration.

Random reference velocity components in $\Delta \dot{\underline{S}}$ from $\widehat{C}^N_{N*}\, \delta \underline{v}^{N*}_{Ref}$ are easily modeled in the integrated velocity Kalman alignment filter using the considered variable approach discussed earlier for the velocity matching type measurement. The $\delta \underline{v}^{N*}_{Ref}$ error in $\Delta \dot{\underline{S}}$ would be treated as part of error state vector x, without estimation in $\hat{\underline{x}}$ of (1), but with inclusion in the covariance matrix P of (2). The $\widehat{A}$ matrix in (2) would be formatted to include $\widehat{C}^N_{N*}$ coupling of the $\delta \underline{v}^{N*}_{Ref}$ covariance into the $\Delta \dot{\underline{S}}$ covariance rate elements of $\dot{P}$. The Kalman gain matrix K_n would be calculated as in (2), with the computed $\delta \underline{v}^{N*}_{Ref}$ rows then set to zero, thereby nullifying $\delta \underline{v}^{N*}_{Ref}$ estimation. The $\delta \underline{v}^{N*}_{Ref}$ error would be modeled in the generic (55) $\dot{\underline{x}}$ format as an easily defined correlated $\delta \dot{\underline{v}}^{N*}_{Ref}$ dynamic process, e.g., first order Markov for each $\delta \underline{v}^{N*}_{Ref}$ component i:

$$\delta \dot{v}^{N*}_{Ref_i} = \widehat{A}_{vRef_i}\, \delta v^{N*}_{Ref_i} + \widehat{G}_{PRef/i}\, n_{PvRef/i} \tag{58}$$

with a selected correlation time embedded in (58) as the negative reciprocal of $\widehat{A}_{vRef_i}$ for each element i. Treating $\delta \underline{v}^{N*}_{Ref}$ as an un-estimated part of error state vector x (with $\delta \dot{v}^{N*}_{Ref_i}$ from (58) in $\dot{\underline{x}}$), the Q_P process noise matrix in (2) for $\dot{P}$ would be structured to include $\underline{n}_{P_{vRef}}$ white noise densities, with the densities set to achieve a specified steady $\delta \underline{v}^{N*}_{Ref}$ variance [2 - Sect.16.2.3.1]. The $\widehat{G}_P$ process noise coupling matrix for $\dot{P}$ in (2) would be structured to include $\widehat{G}_{P_{vRef}}$ coupling of $\underline{n}_{P_{vRef}}$ into $\dot{P}$, and the $\widehat{A}$ error state dynamic coupling matrix would include $\widehat{A}_{vRef}$ coupling of $\delta\underline{v}_{Ref}^{N*}$ into $\dot{P}$.

VELOCITY MATCHING VERSUS INTEGRATED VELOCIY MATCHING

Integrated velocity matching has some advantages compared with the velocity matching measurement approach [2 - Sect. 15.2.2.3] that can best be ascertained by their observation equations in a pure estimation Kalman alignment configuration. For a Kalman estimator, the control vector in (1) is not utilized, and the Kalman filter would estimate system errors as in (1), but without estimated error corrections. To further simplify the measurement approach comparison, it is expedient to approximate the uncontrolled horizontal velocity error $\Delta\underline{v}_{INS\,H}^{N}$ as having little change from its initial value $\Delta\underline{v}_{INS\,H/0}^{N}$. An analytical expression for $\Delta\underline{v}_{INS\,H/0}^{N}$ derives from the initial estimated version of $\hat{\underline{v}}_{INS\,H}^{N}$ in (15):

$$\hat{\underline{v}}_{INS\,H/0}^{N} = \left[\widehat{C}_{N^*0}^{N}\, \hat{\underline{v}}_{Ref\,0}^{N*} + \widehat{C}_{B0}^{N} \left(\hat{\underline{\omega}}_{IB0}^{B} \times \hat{\underline{l}}_{0}^{B} \right) \right]_H \tag{59}$$

The $\Delta\underline{v}_{INS\,H/0}^{N}$ error is derived as (59) minus the equivalent $\underline{v}_{INS\,H/0}^{N}$ from (15) using error definitions in (24):

$$\Delta\underline{v}_{INS\,H/0}^{N} = \Delta C_{N*}^{N}\, \hat{\underline{v}}_{Ref\,H/0}^{N*} + \cdots \tag{60}$$

Approximating $\Delta\underline{v}_{INS\,H}^{N}$ during Kalman alignment by (60), <u>velocity matching</u> observation equation (26) becomes in terms of system errors:

$$\widehat{\underline{M}}_n \approx -\Delta C_{N*}^{N} \left(\hat{\underline{v}}_{Ref\,H/n}^{N*} - \hat{\underline{v}}_{Ref\,H/0}^{N*} \right) + \left(C_B^N \right)_{H/n} \left(\underline{\omega}_{IB_n}^{B} \times \underline{l}_{Flex_n}^{B} \right) + \left(C_B^N \right)_{H/n} \dot{\underline{l}}_{Flex_n}^{B} + \cdots \tag{61}$$

Similarly, (41) with the integral of (45) gives for the <u>integrated velocity matching</u> observation equation:

$$\widehat{\underline{M}}_n \approx -\Delta C_{N*}^{N} \int_0^{t_n} \left(\hat{\underline{v}}_{Ref\,H}^{N*} - \hat{\underline{v}}_{Ref\,H/0}^{N*} \right) dt + C_{B\,H/n}^{N}\, \underline{l}_{Flex_n}^{B} + \cdots \tag{62}$$

Equations (61) and (62) illustrate how lever-arm flexing impacts the observation for the two alignment approaches, both representing measurement noise on the Kalman filter observation input. The form of (61) versus (62) benefits the integrated velocity measurement approach in estimating lever-arm flexing measurement noise for R_M in (2), and in establishing maneuver requirements to achieve a specified heading determination accuracy.

Estimating lever-arm flexing measurement noise for integrated velocity matching entails estimating the magnitude of $\underline{l}_{Flex}^{B}$ in (62) for R_M in (2) under dynamic flight conditions. (In some cases this might be by intuitive judgment of the Kalman filter system design engineer based on a general understanding of the user vehicle in its operational environment,) In contrast, for a velocity matching measurement, estimates of both $\underline{l}_{Flex}^{B}$ and $\dot{\underline{l}}_{Flex}^{B}$ magnitude in (61) are required for R_M, the latter typically having a larger impact on Kalman alignment performance. Flexing rate $\dot{\underline{l}}_{Flex}^{B}$ is a function of both $\underline{l}_{Flex}^{B}$ magnitude and corresponding excited structural bending mode shapes/frequencies that generate $\dot{\underline{l}}_{Flex}^{B}$ proportional to the $\underline{l}_{Flex}^{B}$ mode amplitudes. In general, flexing mode frequency responses are more difficult to estimate than $\underline{l}_{Flex}^{B}$ amplitude, adding a higher degree of uncertainty in estimating $\dot{\underline{l}}_{Flex}^{B}$ magnitude for measurement noise matrix R_M.

Maneuver requirements for determining heading to a specified $\Delta C_{N*_{Required}}^{N}$ accuracy can be ascertained from the $\Delta C_{N*_{Required}}^{N}$ to measurement-noise ratio in (61) and (62). For a velocity matching measurement, the measurement-noise/heading-accuracy ratio in (61) for the dominant flexing rate noise term is symbolically $\left|\dot{\underline{l}}_{Flex_n}^{B}\right| / \left|\Delta C_{N*_{Required}}^{N}\right|$, which sets requirements on the $\left(\hat{\underline{v}}_{Ref_{H/n}}^{N*} - \hat{\underline{v}}_{Ref_{H/0}}^{N*}\right)$ maneuver in (61) to achieve $\Delta C_{N*_{Required}}^{N}$. For example, for a 0.02 ft estimated flexure amplitude at an oscillation frequency of 2 Hz = 12.6 rad/sec, the corresponding flexure rate $\dot{\underline{l}}_{Flex}^{B}$ amplitude would be 0.02 × 12.6 = 0.25 fps. For a 1 milli-rad heading estimation accuracy requirement, the $\left|\dot{\underline{l}}_{Flex_n}^{B}\right| / \left|\Delta C_{N*_{Required}}^{N}\right|$ ratio would be 0.25 / 0.001 = 250 fps. From (61), this requires a $\left(\hat{\underline{v}}_{Ref_{H/n}}^{N*} - \hat{\underline{v}}_{Ref_{H/0}}^{N*}\right)$ single maneuver of 250 fps to achieve the required heading accuracy in a single Kalman estimation cycle. To reduce the maneuver amplitude, multiple measurements would be processed to average out the flexure noise (e.g., for factor of 10 reduction, 10^2 = 100 measurements (and Kalman cycles) would be required). An added complication is that to maintain the same average trajectory direction during the alignment period, the average of $\hat{\underline{v}}_{Ref_{H/n}}^{N*}$ must approximate $\hat{\underline{v}}_{Ref_{H/0}}^{N*}$, necessitating a change in horizontal velocity magnitude, or executing a lateral oscillatory $\hat{\underline{v}}_{Ref_{H/n}}^{N*}$ trajectory change pattern around $\hat{\underline{v}}_{Ref_{H/0}}^{N*}$ (the so-called dynamic "S" maneuver).

In contrast, for an integrated velocity matching measurement, (62) shows that the measurement-noise/heading-accuracy ratio is symbolically $\left|\underline{l}_{Flex_n}^{B}\right| / \left|\Delta C_{N*_{Required}}^{N}\right|$. For the same

0.02 ft lever arm flexure and 1 milli-rad heading error accuracy, $\left|\underline{l}^{B}_{Flex_n}\right| / \left|\Delta C^{N}_{N* Required}\right|$ would be 20 ft. From (62), this corresponds to a $\int_0^{t_n} \left(\hat{\underline{v}}^{N*}_{Ref_H} - \hat{\underline{v}}^{N*}_{Ref_{H/0}} \right)$ horizontal maneuver requirement of 20 ft to achieve the required heading accuracy from a single Kalman estimate. Processing 10 measurements would reduce this by √10 to 6.3 ft. Note, however, that $\int_0^{t_n} \left(\hat{\underline{v}}^{N*}_{Ref_H} - \hat{\underline{v}}^{N*}_{Ref_{H/0}} \right)$ represents a horizontal <u>position</u> displacement from the nominal trajectory. This maneuver can be achieved with a small lateral turn followed by a turn back to the original trajectory direction after the required position displacement has been achieved (i.e., a single half-cycle of an S maneuver). Alternatively, the horizontal velocity magnitude can be slightly altered until the position change has been achieved, the velocity magnitude then returned to its original value. Continued Kalman estimates can then proceed as desired along the modified trajectory with no additional maneuvering.

The previous discussion demonstrates how measurement noise modeling is simplified, and both maneuver and alignment time requirements are reduced for integrated velocity matching compared with velocity matching Kalman alignment. It should also be noted, that due to other system errors (other than heading) being estimated during Kalman alignment, the previous numerical examples would require additional alignment time and/or maneuver increases to achieve the required accuracy.

KALMAN ALIGNMENT INITIALIZATION

Initialization of navigation parameters at the start of Kalman alignment would be as in [6] for the equation (20) parameters, but including a lever arm correction for initial horizontal velocity $\hat{\underline{v}}^{N}_{INS_H}$ as in (59) with $\hat{C}^{N}_{N*}$ initialized at identity as in [6]. The $\hat{C}^{N}_{B}$ attitude matrix would be initialized to the value at Coarse Leveling completion. The best estimate available would be used for $\hat{\underline{l}}^{B}_{0}$ lever arm initialization (zero would be acceptable if completely unknown), and the integrated velocity matching $\hat{\underline{S}}$ parameter would be initialized at zero. All estimated error state parameters would be initialized at zero as in [6].

Initialization of the covariance matrix P would be as in [6] for the (21) navigation error states with a best estimate for the estimated $\delta \underline{l}^{B}_{0}$ lever arm error covariances, and zero for the integrated velocity matching error state $\Delta \underline{S}$ covariances. If first order Markov processes are used as in (58) to model the $\delta \underline{v}^{N*}_{Ref}$ error components, the associated covariances in P would be initialized at their steady state values [2 - Sect. 16.2.3.1].

TRANSITIONING TO NAVIGATION AT KALMAN ALIGNMENT COMPLETION

Kalman alignment completion is declared when the covariance of the (21) heading error parameter $\Delta sin\beta$ falls below a specified level. Then the basic navigation parameters would be initialized for the next phase of either free-inertial or Kalman-aided-inertial navigation. Two methods can be considered in this regard; initialization based on maintaining the N frame at its estimated attitude relative to the B frame in the $\hat{C}_B^N$ matrix, or initialization based on N frame alignment with the reference N^* frame (or with the reference N^{**} frame if using an Appendix A augmentation approach). For either case, navigation mode processing following initialization would then proceed using traditional inertial navigation integration operations, e.g., [2 - Tables 5.6-1, 7.5-1 & 8.4-1] with (1) and (2) as appropriate.

Initialization For Subsequent Free-Inertial Navigation

When the desired navigation mode N frame is defined to be at its alignment completion orientation relative to the B frame, $\hat{C}_B^N$ and $\hat{\underline{v}}_{INS}^N$ would be initialized for the navigation mode at their alignment completion values, altitude would be initialized at the referenced device input altitude h_{Ref} with a $\hat{\underline{l}}_0^B$ lever arm correction, and N frame attitude relative to the E frame (i.e., $\hat{C}_N^E$) would be initialized at the reference input derived $\hat{C}_{N*}^E$ transformed to the N frame:

$$h_{INS} = h_{Ref} + \underline{u}_{ZN}^N \cdot \left(\hat{C}_B^N \, \hat{\underline{l}}_0^B \right) \qquad \hat{C}_N^E = \hat{C}_{N*}^E \left(\hat{C}_{N*}^N \right)^T \tag{63}$$

The $\hat{\underline{l}}_0^B$ term in (63) would be set to its alignment completion value, and $\hat{C}_{N*}^N$ in (63) would be calculated as in (20) using the estimated $\widehat{sin\beta}$ and $\widehat{cos\beta}$ heading parameters at alignment completion. The $\hat{C}_{N*}^E$ matrix in (63) would be computed from reference device input position data using appropriate equivalency formulas, e.g., as [2 - Eq. (4.4.2.1-2)] for latitude, longitude, wander angle reference inputs. (Note - The reference wander angle would be zero for a geographic type N^* frame.) (Another note - The $\hat{C}_N^E$ matrix is also required in the navigation mode as a means for converting N frame data to a known coordinate frame for INS output, e.g., [2 - Eqs. (4.4.2.1-3), (4.1.2-1), (4.1.2-2), & (4.3.1-4)] for latitude, longitude, north/east velocity, roll/pitch/heading. $\hat{C}_N^E$ would be continuously updated during the navigation mode following alignment completion by integrating $\dot{\hat{C}}_N^E$, e.g., [2 - Eq. (4.4.1.1-1)].

When the desired navigation mode N frame is defined to be initially aligned with the N^* frame attitude, $\hat{C}_B^N$ and $\hat{\underline{v}}_{INS}^N$ would be initialized for navigation mode entry at

$$\hat{C}_B^N(+) = \left(\hat{C}_{N^*}^N\right)^T \hat{C}_B^N(-) \qquad \hat{\underline{v}}_{INS}^N(+) = \left(\hat{C}_{N^*}^N\right)^T \hat{\underline{v}}_{INS}^N(-) \tag{64}$$

where

(+), (-) = Values at align completion (-) and at navigation mode entry (+).

The $\hat{C}_N^E$ matrix would be initialized at $\hat{C}_{N^*}^E$, and h_{INS} altitude would be initialized as in (63).

When the desired navigation mode *N* frame is defined to be initially aligned with the N^{**} frame of Appendix A, $\hat{C}_B^N$ and $\hat{\underline{v}}_{INS}^N$ would be initialized at navigation mode entry as

$$\hat{C}_B^N(+) = \left(\hat{C}_{N^*}^N \hat{C}_{N^{**}}^{N^*}\right)^T \hat{C}_B^N(-) \qquad \hat{\underline{v}}_{INS}^N(+) = \left(\hat{C}_{N^*}^N \hat{C}_{N^{**}}^{N^*}\right)^T \hat{\underline{v}}_{INS}^N(-) \tag{65}$$

The $\hat{C}_N^E$ matrix would be initialized at $\hat{C}_{N^{**}}^E$, and h_{INS} altitude would be initialized as in (63).

Initialization For Subsequent Kalman Aided Inertial Navigation

If Kalman alignment is to be followed by a Kalman aided inertial navigation mode, $\hat{C}_B^N$, $\hat{\underline{v}}_{INS}^N$, $\hat{C}_N^E$, and h_{INS} would be initialized as in the previous section for free inertial navigation. In addition, the covariance matrix and initial error states must also be initialized for the aided inertial navigation mode Kalman filter. Two options can be considered: 1. Where the initial navigation mode *N* frame is aligned with the end-of-alignment *N* frame, and 2. When the initial navigation mode *N* frame is aligned with the end-of-alignment *N** (or *N***) frame.

For the former case (maintaining *N* at the end-of-alignment orientation), the $\widehat{\Delta sin\beta}$ estimated error state would be merged with the estimated $\widehat{\gamma_{ZN}}$ component of $\widehat{\underline{\gamma}^N}$ to form an updated version of $\widehat{\underline{\gamma}^N}$. This would include summing the end-of-alignment $\Delta sin\beta$ elements in covariance matrix *P* with those of γ_{ZN} (including summing covariances with other error states). Then, the $\Delta sin\beta$, $\Delta cos\beta$ error states and corresponding rows/columns of *P* would be deleted. Lastly, a vertical velocity error state would be added with, based on $\widehat{v_{ZN}}$ in (20), an estimated initial navigation mode uncertainty of

$$\chi_{\delta v_{ZN\,INS}} = \chi_{\delta v_{ZN^*Ref}} + \underline{u}_{ZN}^N \cdot \left[\hat{C}_B^N\left(\hat{\underline{\omega}}_{IB}^B \times \underline{\chi}_{\delta \underline{l}_0^B}\right)\right] \tag{66}$$

where

$\chi_{\delta v_{ZN\,INS}}$, $\chi_{\delta v_{ZN^*Ref}}$, $\underline{\chi}_{\delta \underline{l}_0^B}$ = $\delta v_{ZN\,INS}$, δv_{ZN^*Ref}, $\delta \underline{l}_0^B$ error uncertainties.

The new $\widehat{\delta v_{ZN\,INS}}$ error state estimate would be initialized at zero in the navigation mode error state vector, and the corresponding elements of the covariance matrix P would be initialized based on (66) as

$$\begin{aligned}
&\mathcal{E}\left(\chi_{\delta v_{ZN\,INS}}{}^2\right) = \mathcal{E}\left(\chi_{\delta v_{ZN^*Ref}}{}^2\right) \\
&+ \left(\underline{u}_{ZN}^N\right)^T \hat{C}_B^N\left(\underline{\hat{\omega}}_{IB}^B\times\right)\mathcal{E}\left(\underline{\chi}_{\delta\underline{l}_0^B}\,\underline{\chi}_{\delta\underline{l}_0^B}{}^T\right)\left[\left(\underline{u}_{ZN}^N\right)^T \hat{C}_B^N\left(\underline{\hat{\omega}}_{IB}^B\times\right)\right]^T \\
&\mathcal{E}\left(\chi_{\delta v_{ZN\,INS}}\,\underline{\chi}_{\delta\underline{l}_0^B}{}^T\right) = \left(\underline{u}_{ZN}^N\right)^T \hat{C}_B^N\left(\underline{\hat{\omega}}_{IB}^B\times\right)\mathcal{E}\left(\underline{\chi}_{\delta\underline{l}_0^B}\,\underline{\chi}_{\delta\underline{l}_0^B}{}^T\right) \\
&\mathcal{E}\left(\underline{\chi}_{\delta\underline{l}_0^B}\,\chi_{\delta v_{ZN\,INS}}\right) = \left[\mathcal{E}\left(\underline{\chi}_{\delta\underline{l}_0^B}\,\chi_{\delta v_{ZN\,INS}}\right)\right]^T
\end{aligned} \qquad (67)$$

The $\mathcal{E}\left(\underline{\chi}_{\delta\underline{l}_0^B}\,\underline{\chi}_{\delta\underline{l}_0^B}{}^T\right)$ term in (67) would be set to the equivalent in the end-of-alignment covariance matrix P. The $\mathcal{E}\left(\chi_{\delta v_{ZN^*Ref}}{}^2\right)$ term would be set to an estimate based on knowledge of reference navigation device accuracy.

For the case when the navigation mode N frame is aligned with the end-of-alignment N^* (or N^{**}) frame, the previous initialization process would also apply in addition to a transformation of the estimated error state and covariance matrix elements from N to N^* (or N**). The former is straightforward but the latter can be complicated regarding correlations with other error states. To avoid this complexity, maintaining the N frame orientation at its end-of-alignment value might be preferred.

Another covariance initialization approach that avoids much of the complexity with the previous methods is to treat the wide angle Kalman alignment process as a Coarse Heading determination operation that follows Coarse Leveling, both then comprising a combined Coarse Alignment function. Course Alignment would be terminated when the $\Delta sin\beta$ variance on the P diagonal became small enough to accurately approximate γ_{ZN} with a linearized model (i.e., a variance of 0.017^2 corresponding to an uncertainty of 1 deg = 0.017 rad), hence, compatible with a traditional aided INS linearized Kalman filter. Then initial attitude for the navigation mode would be set to $\hat{C}_B^N$ at Coarse Alignment completion (with appropriate (64) - (65) N^* or N^{**} conversion if desired), the initial γ_{ZN} variance in P would be set to the specified Coarse Heading Kalman alignment completion value, and the initial $\underline{\gamma}_H^N$ variances would be set to the expected <u>Coarse Leveling</u> accuracies, i.e., the same as when initializing the Coarse Heading alignment Kalman filter. All other parameters for the Kalman aided navigation mode would be initialized as they were at Coarse Heading Kalman alignment <u>initiation</u>, the exception being that the $\Delta sin\beta$, $\Delta cos\beta$ error states would not be included, and the γ_{ZN} variance in the covariance matrix would be initialized at its expected Coarse Alignment completion value.

The penalty for the latter approach is the additional time for the navigation mode Kalman filter to converge the $\underline{\gamma}_H^N$ horizontal attitude errors to their alignment completion values, a minor penalty since these errors would be very visible on the Kalman filter observation vector, hence, attacked immediately at navigation mode initiation (with or without maneuvering). The benefit for the latter approach is avoidance of the analytic development required for transitioning the end-of- alignment Kalman filter into the navigation mode version, the associated time for software development/validation, and the possibility for incurring design errors during the development process, particularly if an *N* to *N** conversion is required. An additional advantage is that the combined dual Coarse-Leveling/Coarse-Heading Alignment function is easily interfaced with previously validated Kalman aided inertial navigation software configurations with their individual error modeling approaches.

APPENDIX A - COMPENSATING FOR DIFFERENCES IN *N* AND *N** FRAME HEADING RATES

This article has assumed that both the *N* and *N** frames are rotated about the vertical at the same rate. For cases when reference data is being supplied in a coordinate frame *N*** having a different vertical rotation rate than *N* and *N**, a simple transformation operation can be used to convert the *N*** vector data into the equivalent *N** frame format assumed in the article. The transformation matrix would represent a rotation around the vertical equal to the integrated difference between the *N*** and *N** vertical rotation rates. For example, consider that the *N* and *N** frames are of the locally level wander azimuth type [2 - Sect. 4.5] whose angular rate around the vertical relative to the earth fixed *E* frame is zero, i.e.,

$$\omega_{EN_{ZN}} = 0 \tag{A-1}$$

where

$\omega_{EN_{ZN}}$ = Component of $\underline{\omega}_{EN}$ along the *N* and *N** frame upward Z axes.

Assume that the *N*** frame used for reference vector data transfer to the INS is of the locally level geographic type [2 - Sect. 4.5] in which one of the horizontal axes (e.g., Y) is controlled by $\omega_{EN_{ZN}}$ to point north. Reference [2 - Eq.(4.4.3-5)] shows that for this situation:

$$\omega_{EN_{ZN**}} = \omega_{EN_{YN**}} \; tan \; l \tag{A-2}$$

where

$\omega_{EN_{ZN**}}$ = Component of $\underline{\omega}_{EN**}$ along the *N*** frame upward Z axis.

$\omega_{EN\,YN^{**}}$ = North (Y axis) component of $\underline{\omega}_{EN^{**}}$.

l = Latitude

The transformation matrix for converting vector data from N** to N* coordinates would be about the vertical through an angle equal to the integrated difference between (A-1) and (A-2) [2 - Eq. (3.3.2-6)]:

$$C_{N^{**}}^{N^*} = \int_0^t \dot{C}_{N^{**}}^{N^*} \, dt \qquad \dot{C}_{N^{**}}^{N^*} = C_{N^{**}}^{N^*}\left(\underline{\omega}_{N^*N^{**}}^{N} \times\right) \qquad \underline{\omega}_{N^*N^{**}}^{N} = \omega_{EN\,YN^{**}} \tan l \; \underline{u}_{ZN}^{N} \tag{A-3}$$

Then any vector $\underline{V}$ provided by the reference navigation device in N^{**} coordinates would be converted to the N^* frame, e.g., for (20), using:

$$\underline{V}^{N^*} = C_{N^{**}}^{N^*} \, \underline{V}^{N^{**}} \tag{A-4}$$

Computations using the transformed data would then proceed assuming N and N^* are rotating at the same rate as described in the article.

APPENDIX B - EVALUATING $\int_{t_{n-1}}^{t} \hat{\underline{v}}_{Ref}^{N^*} \, dt$ FROM REFERENCE POSITION DATA

From the $\underline{v}_{Ref}^{E}$ definition in (5):

$$\underline{v}_{Ref}^{E} \equiv \dot{\underline{R}}_{Ref}^{E} \qquad \Delta\underline{R}_{Ref}^{E} \equiv \underline{R}_{Ref}^{E} - \underline{R}_{Ref_{\,n-1}}^{E} = \int_{t_{n-1}}^{t} \dot{\underline{R}}_{Ref}^{E} \, dt = \int_{t_{n-1}}^{t} \underline{v}_{Ref}^{E} \, dt \tag{B-1}$$

in which Δ in this appendix signifies change, not error (as in the main article). Defining:

$$\Delta\underline{R}_{Ref}^{N^*} = C_E^{N^*} \Delta\underline{R}_{Ref}^{E} \tag{B-2}$$

It follows that:

$$\frac{d}{dt}\left(\Delta\underline{R}_{Ref}^{N^*}\right) = \frac{d}{dt}\left(C_E^{N^*} \Delta\underline{R}_{Ref}^{E}\right) = \dot{C}_E^{N^*} \Delta\underline{R}_{Ref}^{E} + C_E^{N^*} \dot{\underline{R}}_{Ref}^{E} = \dot{C}_E^{N^*} \Delta\underline{R}_{Ref}^{E} + \underline{v}_{Ref}^{N^*} \tag{B-3}$$

The $\dot{C}_E^{N^*}$ term in (B-3) is by similarity with (12) and recognizing that N and N* rotate at the same angular rate:

$$\dot{C}_E^{N^*} = -\, C_E^{N^*}\left(\underline{\omega}_{EN^*}^{E} \times\right) = -\, C_E^{N^*}\left(\underline{\omega}_{EN}^{E} \times\right) \tag{B-4}$$

Then (B-3) becomes

$$\frac{d}{dt}\left(\Delta \underline{R}_{Ref}^{N*}\right) = - C_E^{N*}\left(\underline{\omega}_{EN}^{E}\times\right)\Delta \underline{R}_{Ref}^{E} + \underline{v}_{Ref}^{N*} = - \underline{\omega}_{EN}^{N*}\times\Delta \underline{R}_{Ref}^{N*} + \underline{v}_{Ref}^{N*} \tag{B-5}$$

Integrating (B-5) finds at time t:

$$\Delta \underline{R}_{Ref}^{N*} = -\int_{t_{n-1}}^{t} \underline{\omega}_{EN}^{N*}\times\Delta \underline{R}_{Ref}^{N*}\, dt + \int_{t_{n-1}}^{t} \underline{v}_{Ref}^{N*}\, dt \tag{B-6}$$

Rearranging (B-6) and substituting $\Delta \underline{R}_{Ref}^{E}$ from (B-1) with the $\Delta \underline{R}_{Ref}^{N*}$ definition from (B-2) then obtains for $\int_{t_{n-1}}^{t} \hat{\underline{v}}_{Ref}^{N*}\, dt$ at $t = t_n$

$$\int_{t_{n-1}}^{t_n} \underline{v}_{Ref}^{N*}\, dt = \left(C_E^{N*}\right)_n\left(\underline{R}_{Ref\,n}^{E} - \underline{R}_{Ref\,n\text{-}1}^{E}\right) + \int_{t_{n-1}}^{t_n} \underline{\omega}_{EN}^{N*}\times\left[C_E^{N*}\left(\underline{R}_{Ref}^{E} - \underline{R}_{Ref\,n\text{-}1}^{E}\right)\right] dt \tag{B-7}$$

APPENDIX C - ESTIMATED ERROR STATE DYNAMIC EQUATIONS FOR VELOCITY MATCHING ALIGNMENT KALMAN FILTER

$$\begin{gathered} \frac{d}{dt}\widehat{\Delta sin\beta} = 0 \qquad \frac{d}{dt}\widehat{\Delta cos\beta} = 0 \\ \widehat{\Delta C_{N*}^{N}} = \widehat{\Delta sin\beta}\left(\underline{u}_{ZN}^{N}\times\right) - \widehat{\Delta cos\beta}\left(\underline{u}_{ZN}^{N}\times\right)\left(\underline{u}_{ZN}^{N}\times\right) \end{gathered} \tag{C-1}$$

$$\begin{gathered} \widehat{\Delta\underline{\omega}_{IE\,H}^{N}} = \widehat{\Delta C_{N*}^{N}}\,\hat{\underline{\omega}}_{IE\,Ref}^{N*} \qquad \widehat{\Delta\underline{\omega}_{EN\,H}^{N}} = \widehat{\Delta C_{N*}^{N}}\,\hat{\underline{\omega}}_{EN\,Ref}^{N*} \\ \widehat{\Delta\underline{\omega}_{IN}^{N}} = \widehat{\Delta\underline{\omega}_{IE\,H}^{N}} + \widehat{\Delta\underline{\omega}_{EN\,H}^{N}} \end{gathered} \tag{C-2}$$

$$\begin{gathered} \frac{d}{dt}\widehat{\delta K_{Scal/Mis}} = 0 \qquad \frac{d}{dt}\widehat{\delta \underline{K}_{Bias}} = 0 \\ \widehat{\delta\underline{\omega}_{IB}^{B}} = \widehat{\delta K_{Scal/Mis}}\,\widehat{\underline{\omega}_{IB}^{B}} + \widehat{\delta\underline{K}_{Bias}} \end{gathered} \tag{C-3}$$

$$\dot{\widehat{\underline{\gamma}_{Lin}^{N}}} = -\hat{C}_B^N\,\delta\hat{\underline{\omega}}_{IB}^{B} - \widehat{\underline{\omega}_{IN}^{N}}\times\widehat{\underline{\gamma}_{Lin}^{N}} + \widehat{\Delta\underline{\omega}_{IN\,H}^{N}} \tag{C-4}$$

$$\begin{gathered} \frac{d}{dt}\widehat{\delta L_{Scal/Mis}} = 0 \qquad \frac{d}{dt}\widehat{\delta \underline{L}_{Bias}} = 0 \\ \widehat{\delta\underline{a}_{SF}^{B}} = \widehat{\delta L_{Scal/Mis}}\,\hat{\underline{a}}_{SF}^{B} + \widehat{\delta\underline{L}_{Bias}} \end{gathered} \tag{C-5}$$

$$\widehat{\delta v_{ZN}} = \underline{u}_{ZN}^{N} \cdot \left[\hat{C}_{B}^{N} \left(\underline{\hat{\omega}}_{IB}^{B} \times \widehat{\delta \underline{l}_{0}^{B}} \right) \right] \qquad \dot{\widehat{\delta \underline{l}_{0}^{B}}} = 0 \qquad \widehat{\delta \underline{g}_{P}^{N}} = \widehat{\Delta C_{N*}^{N}} \; \underline{\hat{g}}_{P_{Ref}}^{N*} \qquad \text{(C-6)}$$

$$\begin{aligned} \Delta \dot{\hat{\underline{v}}}_{INS_{Lin/H}}^{N} &= \hat{C}_{B_H}^{N} \; \widehat{\delta \underline{a}_{SF}^{B}} + \left(\widehat{\underline{a}_{SF}^{B}} \times \right)_{H} \widehat{\underline{\gamma}_{Lin}^{N}} + \widehat{\delta \underline{g}_{P\,H}^{N}} \\ &- \left(\underline{u}_{ZN}^{N} \times \right)_{H} \left[\begin{array}{c} \hat{v}_{ZN} \left(\widehat{\Delta \underline{\omega}_{EN_H}^{N}} + 2\, \widehat{\Delta \underline{\omega}_{IE_H}^{N}} \right) + \widehat{\delta v_{ZN}} \left(\underline{\hat{\omega}}_{EN}^{N} + 2\, \underline{\hat{\omega}}_{IE}^{N} \right) \\ + \left(\widehat{\omega_{IN_{ZN}}} + \widehat{\omega_{IE_{ZN}}} \right) \widehat{\Delta \underline{v}_{INS_{Lin/H}}^{N}} \end{array} \right] \end{aligned} \qquad \text{(C-7)}$$

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CHANGES SINCE ORIGINAL APRIL 17, 2015 PUBLICATION

Eq. (20) - Clarified $\underline{\hat{\omega}}_{EN}^{N*}$ definition equation.

Pg. 10 - Clarified definitions at top of page for Eq. (20) change.

Eq. (37) - Changed based on revised Appendix B Eq. (B-7).

Appendix B - Clarified and updated text/equations following Eq. (B-3) for N and N* rates being the same, to correct a transpose error in Eq. (B-7), and to have the Eq. (B-7) result a function of N* frame parameters provided by the reference navigation device.