GENERATING STRAPDOWN SPECIFIC-FORCE/ANGULAR-RATE FOR SPECIFIED ATTITUDE/ POSITION VARIATION FROM A REFERENCE TRAJECTORY

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ABSTRACT

A reference trajectory is commonly used for inertial navigation system (INS) performance evaluation, depicting the attitude/velocity/position profile of a reference source in a moving vehicle (land, sea, or air-borne), and generated by direct previous INS recordings or simulation. Since the advent of Kalman filter INS aiding, it has been typically required to also generate a precision "variation trajectory" from the reference, depicting angular-rate/specific-force and corresponding attitude/velocity/position navigation state at a specified linear-position/angular-orientation relative to the reference. The reference might represent a "master" reference navigation state at a particular location in a moving vehicle, the variation trajectory angular-rate/specific-force would represent inputs to simulated gyros/accelerometers driving the inertial navigation solution in a simulated Kalman aided INS. Comparing the master and aided INS outputs would form the Kalman filter "measurement" input, the output then used to update the INS being aided. The variation trajectory data would also be used to represent the "true" navigation state at the offset location for aided INS performance comparison.

This article analytically derives a new exact formulation for variation trajectory angular-rate/specific-force and the corresponding navigation state as a function of the reference trajectory navigation state and specified attitude/position offsets thereof. The new approach is designed to eliminate unrealistic velocity oscillations created with previous approaches when applying the offset constraints. For performance evaluation, an analytical test example is defined, then used to generate/compare new versus previous variation trajectory solutions relative to the theoretical test example. Results are analytically presented at successive trajectory data update cycles so that detailed performance characteristics can be clearly exposed.

INTRODUCTION

An important element in strapdown INS performance analysis is a reference trajectory used to generate simulated strapdown gyro/accelerometer signals with corresponding navigational attitude/velocity/position for INS software validation and covariance simulation routines. Developing a trajectory that replicates a realistic scenario is a complex computer design process involving maneuver shaping, results prediction, and trajectory generation using previously developed design aids, e.g., [1, Sects. 17.1, 17.2.1, & 17.2.2]. In general, the trajectory generation operation is achieved by processing simulated gyro/accelerometer signals through a precision set of strapdown inertial navigation integration algorithms designed to provide an exact navigation solution response. The trajectory design operation must also assure that the resulting navigation solution properly depicts realistic performance characteristics.

Since the evolution of Kalman aiding for real-time INS calibration/performance-updating, the Kalman filter design process has commonly used a “reference trajectory” to simulate an aiding reference navigation input to the Kalman filter “measurement”. A “variation” trajectory is then used to simulate the INS being aided, providing simulated INS gyro/accelerometer inputs (angular-rate/specific-force), their outputs then processed by aided INS integration routines. Computed aided INS navigation data is provided for user output and “feedback” to the Kalman measurement for comparison with the aiding reference data.

Design of a variation trajectory requires the merging of reference trajectory inputs with variation trajectory constraints to generate a precision solution that realistically represents the navigation state at the aided INS location subject to specified constraints. The variation trajectory constraints might specify a specified offset in aided INS position location in the vehicle relative to the reference navigation data location (including potential vehicle bending effects), and a specified angular orientation of the aided INS relative to reference attitude (including relative angular motion, e.g., simulating a scanning platform).

Ref. [1, Sect. 17.2.3] analytically describes how a variation trajectory can be constructed from a reference trajectory to meet specified attitude/position constraints. However, users who have implemented that approach have reported unusual unexplainable performance anomalies. Even though results exactly satisfied the specified attitude/position constraints, the corresponding angular-rate/specific-force/velocity profile contained unexpected sustained high-frequency oscillations triggered by maneuvers. This article analytically reconstructs the sustained oscillations experienced, and develops an alternate approach for elimination.

The problem with the original [1] approach (identified herein as “Method 1”) stems from the lack of velocity control when generating angular-rate/specific-force (<u>two</u> vector quantities) to meet <u>two</u> specified attitude/position constraints. This is a general problem of defining two vector parameters (angular-rate/specific-force) to satisfy three constraints, attitude/position and an accompanying velocity without unrealistic oscillations. The problem is solved in this article using a different design approach (“Method 2”): Specifying that the attitude constraint still be met at each trajectory update cycle (as with Method 1), but adding an additional velocity constraint that when coupled with the position constraint, is satisfied at alternate (every other) update cycle. The result provides specific-force/angular-rate/attitude/velocity/position at each update cycle, that meets the attitude constraint at each update cycle, and the velocity/position constraints at alternate (“even”) update cycles. Subsequent analysis of Method 2 performance then centers on the realism of the resulting solution, particularly the velocity/position at alternate “odd” update cycles where no velocity/position constraints have been imposed.

The article begins with a general set of exact attitude/velocity/position updating algorithms for variation trajectory response to integrated angular-rate/specific-force increments. Attitude/position constraints are then specified relative to reference trajectory attitude/position. Following is a derivation mimicking the [1] approach, solving for angular-rate/specific-force to meets the attitude/position constraints at each update cycle. The Method 2 approach is then described showing the form of the added velocity constraint, then solving for integrated angular-rate/specific-force increments at each update cycle to meet the attitude constraint at each update cycle and the velocity/position constraints at alternate update cycles.

To analytically evaluate and compare Method 1 and 2 performance, a continuous form sample test example is analytically defined for the reference trajectory with attitude/position constraints, and solved on a continuous basis to form a variation trajectory theoretical trajectory. The test example defines a constant reference velocity plus a sequence of two different constant angular rate maneuvers, with the variation location at a fixed lever arm relative to the reference position. The analytical result defines a constant tangential-velocity/centripetal-acceleration proportional to lever-arm displacement during each constant angular rate segment, but requiring impulsive specific-force at the angular rate change junctions to instantaneously change the associated tangential velocity. For the Method 1 and 2 test example solutions, integrated specific-force increments of finite constant amplitude are used, each spanning an update cycle. As a result, the angular rate changes generate a finite time duration transient in place of the theoretical solution impulse. For Method 1, each transient produces a sustained specific-force/velocity oscillation. For Method 2, the solution matches the theoretical result following the transient (i.e., without sustaining oscillations).

Method 2, however, still contains a small bounded oscillation effect produced by specific-force reaction to balance gravity. That effect remains during constant rotation periods. It is produced by the requirement to meet the velocity/position constraint at alternate (even) update cycles using two sets of specific-force (an odd cycle set followed by an even cycle set). A difference in specific-force gravity balancing (from the theoretical solution) thereby occurs at odd update cycles. The result is a small difference in variation trajectory specific-force from the theoretical solution. The article includes a section showing how the gravity oscillation effect can be effectively removed. The method is by modifying the reference trajectory to contain a matching gravity oscillation. Then, when forming a simulated Kalman filter input "measurement", comparison between the variation and reference trajectories will cancel the gravity oscillation effect.

Preparation of the Method 1 and 2 analytical test model responses was complicated. To simplify the article, only the final test example results are presented here with derivation details provided as Appendices G and H in a separate article [2] directly accessible at http://www.strapdownassociates.com/Appendices%20F,%20G,%20H%20to%20Variation%20Trajectory%20Generator.pdf.

NOTATION

General Notation

$\underline{V}$ = Arbitrary vector without specific coordinate frame component definition.

$\underline{V}^A$ = Column matrix with elements equal to general vector $\underline{V}$ projections on general coordinate frame A axes.

$\left(\underline{V}^{A}\times\right)$ = Cross-product (or skew symmetric) form of $\underline{V}^{A}$ defined such that for the cross-product of $\underline{V}$ with another arbitrary vector $\underline{W}$ in the general *A* frame: $\underline{V}^{A}\times\underline{W}^{A}=\left(\underline{V}^{A}\times\right)\underline{W}^{A}$.

C_{A}^{D} = Generalized direction cosine matrix that transforms vectors from general coordinate frame *A* to general coordinate frame *D* (i.e., $\underline{V}^{D}=C_{A}^{D}\underline{V}^{A}$).

Coordinate Frames

B = "Body" coordinate frame aligned with orthogonal strapdown inertial sensor axes fixed in the rotating body.
$B(t)$ = Frame *B* at time *t*.
E = Earth frame fixed to the rotating earth.
E_0 = Inertial non-rotating inertial frame aligned with *E* at trajectory start time $t = 0$.
I = General inertially non-rotating coordinate frame.
N = Coordinate frame aligned with locally level navigation coordinates having one axis vertical.

Trajectory Generator Update Cycle Indices

m = Trajectory generator update cycle index ($m=0$ at trajectory start time $t = 0$).
n = Trajectory generator even (or alternate) update cycle index (i.e., $m=2n$).
Important Note: Each cycle index **subscript** identifies the ***m* cycle** time instant value for that parameter (e.g., subscript 2*n* indicates a parameter value as cycle $m=2n$, and 2*n*-1 indicates a parameter value at cycle $m=2n-1$.

Trajectory Type Subscripts

ref = Parameter or coordinate frame identifier for the variation trajectory.
var = Parameter or coordinate frame identifier for the variation trajectory.

Parameter Definitions

Parameters are listed next in alphabetical order with Greek letters ordered using the English translation (i.e., Delta Δ under D, mu μ under m, omega ω under o, phi ϕ under p, upsilon υ under u). Parameters used exclusively in the appendices are defined separately in the appendices where they appear.

$\underline{a}_{SF}^{B}$ = Specific force acceleration vector of the rotating body (that would be measured by strapdown accelerometers attached to the rotating body and aligned with body axes).
$C_{A_m}^{D}$ = Direction cosine matrix C_{A}^{D} at the end of update cycle *m*.

$\Delta\underline{\alpha}_m^B$ = Integral over an m cycle of B frame measured inertial angular rate $\underline{\omega}^B$ (that would be measured by strapdown gyros attached to the rotating body and aligned with body axes, i.e., $\Delta\underline{\alpha}_m^B = \int_{t_{m-1}}^{t_m} \underline{\omega}^B dt$.

$\Delta\underline{\alpha}_{var}^{Bvar}$ = Particular value of $\Delta\underline{\alpha}_m^B$ defined for the sample to be constant for $0 \geq m > -9$.

$\Delta\underline{\alpha}'^{Bvar}_{var}$ = Particular value of $\Delta\underline{\alpha}_m^B$ defined for the sample to be constant for $m > 0$.

$\Delta\underline{R}_g(t_m)$ = Change in $\underline{R}$ over the t_{m-1} to t_m time interval caused by $\underline{g}$ gravitational

$\Delta\underline{R}_{SF}(t_m)$ = Change in $\underline{R}$ over the t_{m-1} to t_m time interval caused by specific force $\underline{a}_{SF}$ acceleration.

$\Delta\underline{\upsilon}_m^B$ = Integral over an m cycle of B frame measured specific-force (acceleration) $\underline{a}_{SF}^B$ (i.e., $\Delta\underline{\upsilon}_m^B = \int_{t_{m-1}}^{t_m} \underline{a}_{SF}^B \, dt$).

$\Delta\underline{V}_g(t_m)$ = Change in $\underline{V}$ over the t_{m-1} to t_m time interval caused by caused by $\underline{g}$ gravitational acceleration.

$\Delta\underline{V}_{SF}(t_m)$ = Change in $\underline{V}$ over the t_{m-1} to t_m time interval caused by specific force $\underline{a}_{SF}$ acceleration.

$\underline{g}$ = Earth's mass attraction gravity vector (relative to earth's center) at trajectory position location $\underline{R}$.

$\underline{g}_{avg}$ = Constant average approximation of $\underline{g}$ to simplify the test example model.

I = Identity matrix.

$\underline{l}$ = Specified position displacement vector component of $\underline{S}$.

$\underline{\omega}^B$ = Rotating body angular rate vector relative to non-rotating space (that would be measured by strapdown gyros attached to the body and aligned with rotating body axes).

$\underline{R}$ = Position vector from earth's center to the trajectory designated position location ("navigation center").

$\underline{R}_m$ = Position vector $\underline{R}$ at the end of update cycle m.

$\underline{S}$ = Specified vector displacement of $\underline{R}_{var}$ relative to $\underline{R}_{ref}$.

$\underline{s}$ = Specified additional displacement vector component of $\underline{S}$.

t = Elapsed time from the start of a trajectory.

t_m = Time t at the end of trajectory update cycle m.

T_m = Time interval from t_{m-1} to t_m (assumed constant for this article).

$\underline{V}$ = Velocity of trajectory position relative to non-rotating inertial space defined as the time rate of change of position evaluated in inertially non-rotating E_0 coordinates: $\underline{V}^{E_0} \equiv \dfrac{d\underline{R}^{E_0}}{dt}$.

$\underline{V}_m$ = Velocity vector $\underline{V}$ at the end of update cycle m.

BASIC INERTIAL NAVIGATION OPERATIONS

The general equations of kinematic motion for a rotating body travelling in non-rotating inertial space are provided by the fundamental Newtonian vector expressions [3, Eqs. (1) & (2)]:

$$
\begin{gathered}
\dot{C}_{B(t)}^{B_{m-1}} = C_{B(t)}^{B_{m-1}} \left(\underline{\omega}^B \times \right) \qquad C_{B(t)}^{B_{m-1}} = I + \int_{t_{m-1}}^{t} \dot{C}_{B(t)}^{B_{m-1}} \, dt \\
C_{B_m}^{I} = C_{B_{m-1}}^{I} C_{B(t_m)}^{B_{m-1}} \\
\Delta \underline{\dot{V}}_{SF}^{B_{m-1}} = C_{B(t)}^{B_{m-1}} \underline{a}_{SF}^{B} \qquad \Delta \underline{V}_{SF}^{B_{m-1}}(t) = \int_{t_{m-1}}^{t} \Delta \underline{\dot{V}}_{SF}^{B_{m-1}} \, dt \qquad \Delta \underline{V}_{g}^{I}(t) = \int_{t_{m-1}}^{t} \underline{g}^{I} \, dt \\
\underline{V}_{m}^{I} = \underline{V}_{m-1}^{I} + C_{B_{m-1}}^{I} \Delta \underline{V}_{SF}^{B_{m-1}}(t_m) + \Delta \underline{V}_{g}^{I}(t_m) \\
\Delta \underline{\dot{R}}_{SF}^{B_{m-1}} = \Delta \underline{V}_{SF}^{B_{m-1}} \qquad \Delta \underline{R}_{SF}^{B_{m-1}}(t) = \int_{t_{m-1}}^{t} \Delta \underline{\dot{R}}_{SF}^{B_{m-1}} \, dt \qquad \Delta \underline{R}_{g}^{I}(t) = \int_{t_{m-1}}^{t} \Delta \underline{V}_{g}^{I} \, dt \\
\underline{R}_{m}^{I} = \underline{R}_{m-1}^{I} + \underline{V}_{m-1}^{I} T_m + C_{B_{m-1}}^{I} \Delta \underline{R}_{SF}^{B_{m-1}}(t_m) + \Delta \underline{R}_{g}^{I}(t_m)
\end{gathered}
\tag{1}
$$

Eqs. (1) are the basis for design of a strapdown inertial navigation system (INS) in which angular-rate/specific force ($\underline{\omega}$, $\underline{a}_{SF}$) are measured for input, and attitude, velocity, position (C_B^I , $\underline{V}^I$, $\underline{R}^I$) are computed in the non-rotating inertial *I* frame for output. In most INSs, (1) is defined for integration in an intermediate locally level "navigation" *N* frame with velocity defined as the rate of change of position relative to the rotating earth, and position is designated by altitude and angular position in an earth based coordinate frame (e.g., latitude/longitude or the equivalent direction cosine matrix representation). The resulting equations are equivalent to (1), but with additional terms associated with angular velocity of the locally level *N* frame and earth's angular rotation rate (i.e., the so-called centripetal and Coriolis acceleration terms added to the velocity rate equations). Integration of the *N* frame equations, however, yields exactly the same solution as integration of (1), after the results are converted to the *I* frame attitude, velocity, position definitions (and assuming the use of "exact" (error free) integration algorithms).

Trajectory generator simulators have also been based on the integration of (1) (or their equivalent *N* frame version) with one notable difference, for a trajectory generator, angular-rate/specific-force ($\underline{\omega}$, $\underline{a}_{SF}$) are programmed functions that have been previously determined to yield an attitude/velocity/position (C_B^I, $\underline{V}^I$, $\underline{R}^I$) profile representative of user operations. Design of $\underline{\omega}$, $\underline{a}_{SF}$ to generate a representative C_B^I, $\underline{V}^I$, $\underline{R}^I$ profile is a complicated process that can be simplified by a fundamental premise in the trajectory profile structure: to consist of a sequence of trajectory segments, each based on constant $\underline{\omega}$, $\underline{a}_{SF}$ values during successive trajectory update *m* cycles of (1) within the segment.

This article describes the design of a "variation trajectory" generator that produces a trajectory at specified attitude/position variation from an existing "reference" trajectory. The variation trajectory is assumed to be created by a trajectory generator structured as described previously, by a sequence of constant specific force/angular-rate segments, each spanning an *m* cycle of (1). The result (for an I frame representation) will be an exact solution of (1) for each m cycle as described in [3, Eqs. (2) – (5)]:

$$
\begin{aligned}
\Delta\underline{\alpha}_{var_m}^{Bvar} &\equiv \int_{t_{m-1}}^{t_m} \underline{\omega}_{var}^{Bvar} dt \\
G_{Cvar_m} &\equiv I + \frac{\sin\Delta\alpha_{var_m}}{\Delta\alpha_{var_m}}\left(\Delta\underline{\alpha}_{var_m}^{Bvar}\times\right) + \frac{1-\cos\Delta\alpha_{var_m}}{\Delta\alpha_{var_m}^2}\left(\Delta\underline{\alpha}_{var_m}^{Bvar}\times\right)^2 \\
C_{Bvar_m}^{E0} &= C_{Bvar_{m-1}}^{E0} G_{Cvar_m}
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
G_{Vvar_m} &\equiv I + \frac{1-\cos\Delta\alpha_{var_m}^{Bvar}}{\left(\Delta\alpha_{var_m}^{Bvar}\right)^2}\left(\Delta\underline{\alpha}_{var_m}^{Bvar}\times\right) + \frac{1}{\Delta\alpha_{var_m}^2}\left(1-\frac{\sin\Delta\alpha_{var_m}^{Bvar}}{\Delta\alpha_{var_m}^{Bvar}}\right)\left(\Delta\underline{\alpha}_{var_m}^{Bvar}\times\right)^2 \\
\Delta\underline{\upsilon}_{var_m}^{Bvar} &\equiv \int_{t_{m-1}}^{t_m} \underline{a}_{SF\,var}^{Bvar} dt \\
\underline{V}_{var_m}^{E0} &= \underline{V}_{var_{m-1}}^{E0} + C_{Bvar_{m-1}}^{E0} G_{Vvar_m} \Delta\underline{\upsilon}_{var_m}^{Bvar} + \frac{1}{2}\left(\underline{g}_{var_m}^{E0} + \underline{g}_{var_{m-1}}^{E0}\right) T_m
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
G_{Rvar_m} \equiv\; & \frac{1}{2} I + \frac{1}{\left(\Delta\alpha_{var_m}^{Bvar}\right)^2}\left(1-\frac{\sin\Delta\alpha_{var_m}^{Bvar}}{\Delta\alpha_{var_m}^{Bvar}}\right)\left(\Delta\underline{\alpha}_{var_m}^{Bvar}\times\right) \\
& + \frac{1}{\left(\Delta\alpha_{var_m}^{Bvar}\right)^2}\left[\frac{1}{2}-\left(\frac{1-\cos\Delta\alpha_{var_m}^{Bvar}}{\left(\Delta\alpha_{var_m}^{Bvar}\right)^2}\right)\right]\left(\Delta\underline{\alpha}_{var_m}^{Bvar}\times\right)^2 \\
\underline{R}_{var_m}^{E0} =\; & \underline{R}_{var_{m-1}}^{E0} + \underline{V}_{var_{m-1}}^{E0} T_m + C_{Bvar_{m-1}}^{E0} G_{Rvar_m} \Delta\underline{\upsilon}_{var_m}^{Bvar} T_m + \frac{1}{6}\left(\underline{g}_{var_m}^{E0} + 2\,\underline{g}_{var_{m-1}}^{E0}\right) T_m^2
\end{aligned}
\tag{4}
$$

The $\underline{g}_{var}^{E0}$ gravity groupings in (3) and (4) are based on a linearly varying gravity vector over each m cycle as derived in Appendix A. For each m cycle, $\underline{g}_{var}^{E0}$ (specifically, $\underline{g}_{var_m}^{E0}$ at the end of the m cycle) is a function of variation trajectory position vector $\underline{R}_{var_m}^{E0}$, and can be calculated directly from $\underline{R}_{var_m}^{E0}$ (as shown in Appendix B), or as a first order expansion around the reference trajectory gravity vector (as shown in Appendix C).

SPECIFIED VARIATION TRAJECTORY CONSTRAINTS

The variation trajectory is defined to vary from the reference trajectory by defined attitude and position offsets (constraints) at each *m* time point.

The attitude constraint is defined as a specified angular orientation relative to the reference trajectory attitude matrix:

$$C^{E0}_{Bvar_m} = C^{E0}_{Bref_m} \; C^{Bref_m}_{Bvar_m} \tag{5}$$

The position constraint is defined as a specified linear translation relative to the reference trajectory position location:

$$\underline{R}^{E0}_{var_m} = \underline{R}^{E0}_{ref_m} + \underline{S}^{E0}_{m} \tag{6}$$

For flexibility in use, the $\underline{S}^{E0}_{m}$ displacement vector is further defined to be composed of two components, each represented by a different coordinate frame projection:

$$\underline{S}^{E0}_{m} \equiv C^{E0}_{Bvar_m}\left[\left(C^{Bref}_{Bvar_m}\right)^T \underline{l}^{Bref_m} + \underline{s}^{Bvar_m}\right] \tag{7}$$

If the reference trajectory was defined to represent a master INS in an aircraft, the $\underline{l}^{Bref_m}$ vector might represent a lever arm from the master INS location to a second INS location in the aircraft, and $\underline{s}^{Bvar_m}$ could represent an additional displacement of the second INS location when mounted on a controlled rotating platform at instantaneous orientation $C^{Bref}_{Bvar_m}$ relative to the master INS attitude.

For complete flexibility, $C^{Bref}_{Bvar_m}$, $\underline{l}^{Bref_m}$, and $\underline{s}^{Bvar_m}$ can be represented by constant and time varying terms, $C^{Bref}_{Bvar_m}$ changes with time representing (for example) a scanning operation, and the $\underline{l}^{Bref_m}$, $\underline{s}^{Bvar_m}$ variations representing (for example) fixed plus bending oscillations. (Ref. [1, Sect. 17.2.3.2.3] describes how random finite bandwidth lever arm bending effects can be analytically modeled.) For realism in their use, the time change effects can be filtered to eliminate (for example) unrealistically abrupt changes in their movement at programmed model time junctions, e.g., changing the $C^{Bref}_{Bvar_m}$ scan rate from one value to another. Based on [1, Sect. 17.2.1], Appendix D describes how filtering can be easily incorporated in $C^{Bref}_{Bvar_m}$ for smoothing while retaining bounded specified offset characteristics and the traditional orthogonality/normality properties of a direction cosine matrix.

DESIGNING THE VARIATION TRAJECTORY TO MEET THE SPECIFIED CONSTRAINTS

Designing the variation trajectory requires solving for $\Delta\underline{\alpha}_{var_m}^{Bvar}$, $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ in (2) – (4) to meet the (5) – (7) constraints. Once $\Delta\underline{\alpha}_{var_m}^{Bvar}$, $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ are determined, the variation trajectory can be generated by processing (2) – (4) in non-rotating inertial E_0 coordinates (or an equivalent in locally level navigation coordinates, e.g., azimuth wander [1, Sect. 4.5 & Table 5.6-1].

Solving for $\Delta\underline{\alpha}_{var_m}^{Bvar}$ is a straight-forward matrix transformation operation of (2) using constraint (5) for $C_{Bvar_m}^{E0}$. Substituting $C_{Bvar_m}^{E0}$ from (5) into the third equation in (2), then multiplying by the inverse of $C_{Bvar_{m-1}}^{E0}$ (while recognizing that the inverse of a direction cosine matrix equals its transpose), finds G_{Cvar_m}. Identifying $\Delta\underline{\alpha}_{var_m}^{Bvar}$ as a rotation vector (based on its definition in (2) and the assumed constant angular rate $\underline{\omega}_{var}^{Bvar}$ over an *m* cycle for the variation trajectory), G_{Cvar_m} is solved for $\Delta\underline{\alpha}_{var_m}^{Bvar}$ using a direction cosine matrix to rotation vector inversion routine [1, Sect. 3.2.2.2]. Thus, attitude constraint (5) is implemented in the variation trajectory simulator by replacing (2) with

$$
\begin{aligned}
&C_{Bvar_m}^{E0} = C_{Bref_m}^{E0}\, C_{Bvar_m}^{Bref_m} \qquad G_{Cvar_m} = C_{E0_{m-1}}^{Bvar}\, C_{Bvar_m}^{E0} \\
&\Delta\underline{\alpha}_{var_m}^{Bvar} = \text{Rotation Vector Extraction From } G_{Cvar_m}
\end{aligned}
\tag{8}
$$

With $\Delta\underline{\alpha}_{var_m}^{Bvar}$ so determined, G_{Vvar_m} and G_{Rvar_m} (both functions of $\Delta\underline{\alpha}_{var_m}^{Bvar}$) are calculated from their definitions in (3) and (4), then applied to find a $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ specific force that generates a $\underline{R}_{var_m}^{E0}$ position profile meeting the specified position constraints. Two methods have been analyzed for finding $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ that satisfy position constraint (6) - (7):

1. Solving for $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ over each *m* cycle with $\underline{R}_{var_m}^{E0}$ satisfying position constraint (6) – (7), but without specific control of velocity $\underline{V}_{var_m}^{E0}$.

2. Solving for $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ over each *m* cycle so that position $\underline{R}_{var_m}^{E0}$ satisfies (6) - (7) at alternate cycles, and including specific control of velocity $\underline{V}_{var_m}^{E0}$ at the alternate *m* cycles.

Each is discussed next followed by an example for analytically comparing the $\Delta \underline{\upsilon}_{var_m}^{Bvar}$, $\underline{V}_{var_m}^{E0}$, $\underline{R}_{var_m}^{E0}$ profiles generated by each method.

METHOD 1 – SATISFYING THE POSITION CONSTRAINT AT EACH *m* CYCLE

Method 1 is an inertial frame version of the locally level frame equivalent in [1, Eq. (17.2.3.2-24)] whereby $\Delta \underline{\upsilon}_{var_m}^{Bvar}$ is defined by direct inversion of (4):

$$\Delta \underline{\upsilon}_{var_m}^{Bvar} = \left(C_{Bvar_{m-1}}^{E0} \, G_{Rvar_m} \right)^{-1} \begin{bmatrix} \underline{R}_{var_m}^{E0} - \underline{R}_{var_{m-1}}^{E0} - \underline{V}_{var_{m-1}}^{E0} \, T_m \\ -\dfrac{1}{6}\left(\underline{g}_{var_m}^{E0} + 2\, \underline{g}_{var_{m-1}}^{E0} \right) T_m^2 \end{bmatrix} / \, T_m \qquad (9)$$

In (9), $\underline{R}_{var_m}^{E0}$ and $\underline{R}_{var_{m-1}}^{E0}$ are set to the current and previous value specified by position constraint (6) with (7). The previous *m* cycle velocity $\underline{V}_{var_{m-1}}^{E0}$ in (9) would be provided from the past cycle of (3), necessitating that (3) be included as part of $\Delta \underline{\upsilon}_{var_m}^{Bvar}$ determination. The $\underline{g}_{var_m}^{E0}$ gravity terms in (9) are functions of $\underline{R}_{var_m}^{E0}$ specified position constraint (6), and can be calculated using either of the Appendix B or C approaches.

Variation position $\underline{R}_{var_m}^{E0}$ in (9) is calculated in (6) - (7) with $C_{Bvar_m}^{E0}$ from (8) using reference position $\underline{R}_{ref_m}^{E0}$ and attitude $C_{Bref_m}^{E0}$ provided from the reference trajectory. If not directly available in (6) and (8) format, $\underline{R}_{ref_m}^{E0}$, $C_{Bref_m}^{E0}$ can be obtained by conversion of available reference position data (e.g., body frame $Bref$ attitude relative to a locally-level wander azimuth coordinate frame, altitude, and wander azimuth frame position angle, as described in Appendix E). The G_{Rvar_m} term in (9) would be calculated as in (4).

Once $\Delta \underline{\upsilon}_{var_m}^{Bvar}$ is determined, velocity $\underline{V}_{var_m}^{E0}$ and position $\underline{R}_{var_m}^{E0}$ are obtained using (3) and (4). (Note: Since Method 1 specific force is based on meeting position constraint (6) with (7), (6) rather than (4) can be used directly for $\underline{R}_{var_m}^{E0}$). Attitude/velocity/position results can then be output directly in E_0 inertial coordinates, or after conversion to another desired format as required (e.g., Appendix E).

The problem with the Method 1 approach is that even though integrated specific force increment $\Delta \underline{\upsilon}_{var_m}^{Bvar}$ calculated by (9) will correctly generate $\underline{R}_{var_m}^{E0}$ that meets constraint (6), the resulting $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ and $\underline{V}_{var_m}^{E0}$ obtained from $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ in (3), may not look realistic. (This was unexpected by the author when applying [1, Sect. 17.2.3.2] in actual simulations, and which motivated the publication of this article for mitigation.) The Method 1 problem is described in more detail subsequently by test example, showing that even though position constraint (6) is satisfied at each *m* cycle, an unrealistic oscillation is generated in the accompanying $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ and $\underline{V}_{var_m}^{E0}$ solutions.

By adding a velocity constraint to go with position constraint (6), Method 2 (described next) mitigates the Method 1 oscillation effect, but at sacrifice of position constraint (6) only being satisfied at alternate *m* cycles. Method 2 performance is then demonstrated subsequently in this article when applied to the same test example used for Method 1.

METHOD 2 – SATISFYING THE POSITION CONSTRAINT AT ALTERNATE *m* CYCLES

Method 2 uses two sequential cycles of $\Delta\underline{\upsilon}_{var_m}^{Bvar}$, each based on constant rate/acceleration, that meet the (6) specified position constraint on $\underline{R}_{var_m}^{E0}$ at alternate *m* cycles, but also satisfy an additional constraint on $\underline{V}_{var_m}^{E0}$ velocity. The velocity constraint sets $\underline{V}_{var_m}^{E0}$ to the time derivative of position constraint (6) at time instant *m*, with the derivative of $\underline{S}^{E0}$ in (6) approximated as the average change of $\underline{S}^{E0}$ over two *m* cycles:

$$\begin{aligned}\underline{V}_{var_m}^{E0} &= \left(\frac{d\,\underline{R}_{ref}^{E0}}{dt}\right)_m + \left(\frac{d\,\underline{S}^{E0}}{dt}\right)_m \approx \left(\frac{d\,\underline{R}_{ref}^{E0}}{dt}\right)_m + \text{Estimated}\left(\frac{d\,\underline{S}^{E0}}{dt}\right)_m \\ &= \underline{V}_{ref_m}^{E0} + \frac{\underline{S}_{m+1}^{E0} - \underline{S}_{m-1}^{E0}}{2\,T_m}\end{aligned} \qquad (10)$$

Because $\underline{S}_m^{E0}$ is specified by (6) at each *m* cycle, $\underline{S}_{m+1}^{E0}$ in (10) requires a computation of (6) one *m* cycle in advance (to be built into the variation trajectory generator non-real-time structure).

Deriving two *m* cycles of $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ expressions that satisfy (6) and (10) requires the *m*-1 cycle version of (3) and (4) for $\underline{V}_{var_{m-1}}^{E0}$, $\underline{R}_{var_{m-1}}^{E0}$ to go with $\underline{V}_{var_m}^{E0}$, $\underline{R}_{var_m}^{E0}$ in (3) and (4). Permuting subscripts in (4) first finds for $\underline{R}_{var_{m-1}}^{E0}$:

$$\underline{R}^{E0}_{var_{m-1}} = \underline{R}^{E0}_{var_{m-2}} + \underline{V}^{E0}_{var_{m-2}} T_m + C^{E0}_{Bvar_{m-2}} G_{Rvar_{m-1}} \Delta \underline{\upsilon}^{Bvar}_{var_{m-1}} T_m + \frac{1}{6}\left(\underline{g}^{E0}_{var_{m-1}} + 2\, \underline{g}^{E0}_{var_{m-2}} \right) T_m^2 \tag{11}$$

Substituting (11) for $\underline{R}^{E0}_{var_{m-1}}$ in the (4) $\underline{R}^{E0}_{var_m}$ expression obtains

$$\begin{aligned} \underline{R}^{E0}_{var_m} &= \underline{R}^{E0}_{var_{m-2}} + \left(\underline{V}^{E0}_{var_{m-1}} + \underline{V}^{E0}_{var_{m-2}} \right) T_m \\ &+ \left(C^{E0}_{Bvar_{m-2}} G_{Rvar_{m-1}} \Delta \underline{\upsilon}^{Bvar}_{var_{m-1}} + C^{E0}_{Bvar_{m-1}} G_{Rvar_m} \Delta \underline{\upsilon}^{Bvar}_{var_m} \right) T_m \\ &+ \frac{1}{6}\left(\underline{g}^{E0}_{var_m} + 3\, \underline{g}^{E0}_{var_{m-1}} + 2\, \underline{g}^{E0}_{var_{m-2}} \right) T_m^2 \end{aligned} \tag{12}$$

or upon rearrangement:

$$\begin{aligned} &\left(C^{E0}_{Bvar_{m-2}} G_{Rvar_{m-1}} \Delta \underline{\upsilon}^{Bvar}_{var_{m-1}} + C^{E0}_{Bvar_{m-1}} G_{Rvar_m} \Delta \underline{\upsilon}^{Bvar}_{var_m} \right) T_m \\ &= \underline{R}^{E0}_{var_m} - \underline{R}^{E0}_{var_{m-2}} - \left(\underline{V}^{E0}_{var_{m-1}} + \underline{V}^{E0}_{var_{m-2}} \right) T_m - \frac{1}{6}\left(\underline{g}^{E0}_{var_m} + 3\, \underline{g}^{E0}_{var_{m-1}} + 2\, \underline{g}^{E0}_{var_{m-2}} \right) T_m^2 \end{aligned} \tag{13}$$

Similarly, permuting subscripts in (3) finds for $\underline{V}^{E0}_{var_{m-1}}$:

$$\underline{V}^{E0}_{var_{m-1}} = \underline{V}^{E0}_{var_{m-2}} + C^{E0}_{Bvar_{m-2}} G_{Vvar_{m-1}} \Delta \underline{\upsilon}^{Bvar}_{var_{m-1}} + \frac{1}{2}\left(\underline{g}^{E0}_{var_{m-1}} + \underline{g}^{E0}_{var_{m-2}} \right) T_m \tag{14}$$

which when substituted in (3) obtains

$$\begin{aligned} \underline{V}^{E0}_{var_m} &= \underline{V}^{E0}_{var_{m-2}} + C^{E0}_{Bvar_{m-2}} G_{Vvar_{m-1}} \Delta \underline{\upsilon}^{Bvar}_{var_{m-1}} + \frac{1}{2}\left(\underline{g}^{E0}_{var_{m-1}} + \underline{g}^{E0}_{var_{m-2}} \right) T_m \\ &+ C^{E0}_{Bvar_{m-1}} G_{Vvar_m} \Delta \underline{\upsilon}^{Bvar}_{var_m} + \frac{1}{2}\left(\underline{g}^{E0}_{var_m} + \underline{g}^{E0}_{var_{m-1}} \right) T_m \end{aligned} \tag{15}$$

With rearrangement, (15) becomes

$$\begin{aligned} C^{E0}_{Bvar_{m-1}} G_{Vvar_m} \Delta \underline{\upsilon}^{Bvar}_{var_m} + C^{E0}_{Bvar_{m-2}} G_{Vvar_{m-1}} \Delta \underline{\upsilon}^{Bvar}_{var_{m-1}} &= \underline{V}^{E0}_{var_m} - \underline{V}^{E0}_{var_{m-2}} \\ &- \frac{1}{2}\left(\underline{g}^{E0}_{var_m} + 2\, \underline{g}^{E0}_{var_{m-1}} + \underline{g}^{E0}_{var_{m-2}} \right) T_m \end{aligned} \tag{16}$$

A rearranged form of (14) will also prove useful:

$$\underline{V}^{E0}_{var_{m-1}} + \underline{V}^{E0}_{var_{m-2}} = 2\,\underline{V}^{E0}_{var_{m-2}} + C^{E0}_{Bvar_{m-2}}\,G_{Vvar_{m-1}}\,\Delta\underline{\upsilon}^{Bvar}_{var_{m-1}} + \frac{1}{2}\left(\underline{g}^{E0}_{var_{m-1}} + \underline{g}^{E0}_{var_{m-2}}\right)T_m \quad (17)$$

Substituting $\left(\underline{V}^{E0}_{var_{m-1}} + \underline{V}^{E0}_{var_{m-2}}\right)$ from (17) in (12) obtains after rearrangement:

$$\begin{aligned}
&\left(C^{E0}_{Bvar_{m-2}}\,G_{Rvar_{m-1}}\,\Delta\underline{\upsilon}^{Bvar}_{var_{m-1}} + C^{E0}_{Bvar_{m-1}}\,G_{Rvar_m}\,\Delta\underline{\upsilon}^{Bvar}_{var_m}\right)T_m \\
&= \underline{R}^{E0}_{var_m} - \underline{R}^{E0}_{var_{m-2}} - \left(2\,\underline{V}^{E0}_{var_{m-2}} + C^{E0}_{Bvar_{m-2}}\,G_{Vvar_{m-1}}\,\Delta\underline{\upsilon}^{Bvar}_{var_{m-1}}\right)T_m \qquad (18) \\
&\qquad -\frac{1}{6}\left(\underline{g}^{E0}_{var_m} + 6\,\underline{g}^{E0}_{var_{m-1}} + 5\,\underline{g}^{E0}_{var_{m-2}}\right)T_m^2
\end{aligned}$$

which combines into

$$\begin{aligned}
&\left[C^{E0}_{Bvar_{m-2}}\left(G_{Rvar_{m-1}} + G_{Vvar_{m-1}}\right)\Delta\underline{\upsilon}^{Bvar}_{var_{m-1}} + C^{E0}_{Bvar_{m-1}}\,G_{Rvar_m}\,\Delta\underline{\upsilon}^{Bvar}_{var_m}\right]T_m \\
&= \underline{R}^{E0}_{var_m} - \underline{R}^{E0}_{var_{m-2}} - 2\,\underline{V}^{E0}_{var_{m-2}}\,T_m - \frac{1}{6}\left(\underline{g}^{E0}_{var_m} + 6\,\underline{g}^{E0}_{var_{m-1}} + 5\,\underline{g}^{E0}_{var_{m-2}}\right)T_m^2
\end{aligned} \qquad (19)$$

Eqs. (16) and (19) are now in a convenient form to find $\Delta\underline{\upsilon}^{Bvar}_{var_m}$ and $\Delta\underline{\upsilon}^{Bvar}_{var_{m-1}}$. First (16) and (19) are solved for $\Delta\underline{\upsilon}^{Bvar}_{var_m}\,T_m$:

$$\begin{aligned}
\Delta\underline{\upsilon}^{Bvar}_{var_m}\,T_m &= \left(C^{Bvar\,0}_{Bvar_{m-1}}\,G_{Vvar_m}\right)^{-1}\left[\begin{array}{c}
\underline{V}^{E0}_{var_m} - \underline{V}^{E0}_{var_{m-2}} \\
-\frac{1}{2}\left(\underline{g}^{E0}_{var_m} + 2\,\underline{g}^{E0}_{var_{m-1}} + \underline{g}^{E0}_{var_{m-2}}\right)T_m \\
-\,C^{E0}_{Bvar_{m-2}}\,G_{Vvar_{m-1}}\,\Delta\underline{\upsilon}^{Bvar}_{var_{m-1}}
\end{array}\right]T_m \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (20) \\
\Delta\underline{\upsilon}^{Bvar}_{var_m}\,T_m &= \left(C^{E0}_{Bvar_{m-1}}\,G_{Rvar_m}\right)^{-1}\left[\begin{array}{c}
\underline{R}^{E0}_{var_m} - \underline{R}^{E0}_{var_{m-2}} - 2\,\underline{V}^{E0}_{var_{m-2}}\,T_m \\
-\frac{1}{6}\left(\underline{g}^{E0}_{var_m} + 6\,\underline{g}^{E0}_{var_{m-1}} + 5\,\underline{g}^{E0}_{var_{m-2}}\right)T_m^2 \\
-\,C^{E0}_{Bvar_{m-2}}\left(G_{Rvar_{m-1}} + G_{Vvar_{m-1}}\right)\Delta\underline{\upsilon}^{Bvar}_{var_{m-1}}\,T_m
\end{array}\right]
\end{aligned}$$

Equating the $\Delta\underline{\upsilon}^{Bvar}_{var_m}\,T_m$ expressions in (20) yields

$$\left(C_{Bvar_{m-1}}^{Bvar_0}\, G_{Vvar_m}\right)^{-1} \begin{bmatrix} \underline{V}_{var_m}^{E0} - \underline{V}_{var_{m-2}}^{E0} - \frac{1}{2}\left(\underline{g}_{var_m}^{E0} + 2\,\underline{g}_{var_{m-1}}^{E0} + \underline{g}_{var_{m-2}}^{E0}\right) T_m \\ - C_{Bvar_{m-2}}^{E0}\, G_{Vvar_{m-1}}\, \Delta\underline{\upsilon}_{var_{m-1}}^{Bvar} \end{bmatrix} T_m$$

$$= \left(C_{Bvar_{m-1}}^{E0}\, G_{Rvar_m}\right)^{-1} \begin{bmatrix} \underline{R}_{var_m}^{E0} - \underline{R}_{var_{m-2}}^{E0} - 2\,\underline{V}_{var_{m-2}}^{E0}\, T_m \\ -\frac{1}{6}\left(\underline{g}_{var_m}^{E0} + 6\,\underline{g}_{var_{m-1}}^{E0} + 5\,\underline{g}_{var_{m-2}}^{E0}\right) T_m^2 \\ - C_{Bvar_{m-2}}^{E0}\left(G_{Rvar_{m-1}} + G_{Vvar_{m-1}}\right) \Delta\underline{\upsilon}_{var_{m-1}}^{Bvar}\, T_m \end{bmatrix} \tag{21}$$

or upon rearrangement:

$$\begin{aligned} &\underline{R}_{var_m}^{E0} - \underline{R}_{var_{m-2}}^{E0} - 2\,\underline{V}_{var_{m-2}}^{E0}\, T_m - \frac{1}{6}\left(\underline{g}_{var_m}^{E0} + 6\,\underline{g}_{var_{m-1}}^{E0} + 5\,\underline{g}_{var_{m-2}}^{E0}\right) T_m^2 \\ &\qquad - C_{Bvar_{m-2}}^{E0}\left(G_{Rvar_{m-1}} + G_{Vvar_{m-1}}\right) \Delta\underline{\upsilon}_{var_{m-1}}^{Bvar}\, T_m \\ &= C_{Bvar_{m-1}}^{E0}\, G_{Rvar_m} \left(C_{Bvar_{m-1}}^{E0}\, G_{Vvar_m}\right)^{-1} \begin{bmatrix} \underline{V}_{var_m}^{E0} - \underline{V}_{var_{m-2}}^{E0} \\ -\frac{1}{2}\left(\underline{g}_{var_m}^{E0} + 2\,\underline{g}_{var_{m-1}}^{E0} + \underline{g}_{var_{m-2}}^{E0}\right) T_m \\ - C_{Bvar_{m-2}}^{Bvar_0}\, G_{Vvar_{m-1}}\, \Delta\underline{\upsilon}_{var_{m-1}}^{Bvar} \end{bmatrix} T_m \end{aligned} \tag{22}$$

Define

$$A_{m-1} \equiv C_{Bvar_{m-1}}^{E0}\, G_{Rvar_m} \left(C_{Bvar_{m-1}}^{E0}\, G_{Vvar_m}\right)^{-1} \tag{23}$$

with which (22) becomes

$$\begin{aligned} &\underline{R}_{var_m}^{E0} - \underline{R}_{var_{m-2}}^{E0} - 2\,\underline{V}_{var_{m-2}}^{E0}\, T_m - \frac{1}{6}\left(\underline{g}_{var_m}^{E0} + 6\,\underline{g}_{var_{m-1}}^{E0} + 5\,\underline{g}_{var_{m-2}}^{E0}\right) T_m^2 \\ &\qquad - C_{Bvar_{m-2}}^{E0}\left(G_{Rvar_{m-1}} + G_{Vvar_{m-1}}\right) \Delta\underline{\upsilon}_{var_{m-1}}^{Bvar}\, T_m \\ &= A_{m-1} \begin{bmatrix} \underline{V}_{var_m}^{E0} - \underline{V}_{var_{m-2}}^{E0} \\ -\frac{1}{2}\left(\underline{g}_{var_m}^{E0} + 2\,\underline{g}_{var_{m-1}}^{E0} + \underline{g}_{var_{m-2}}^{E0}\right) T_m \\ - C_{Bvar_{m-2}}^{E0}\, G_{Vvar_{m-1}}\, \Delta\underline{\upsilon}_{var_{m-1}}^{Bvar} \end{bmatrix} T_m \end{aligned} \tag{24}$$

Equivalently, after grouping like terms:

$$\left[C^{E0}_{Bvar_{m-2}}\left(G_{Rvar_{m-1}}+G_{Vvar_{m-1}}\right)-A_{m-1}\,C^{E0}_{Bvar_{m-2}}\,G_{Vvar_{m-1}}\right]\Delta\underline{\upsilon}^{Bvar}_{var_{m-1}}\,T_m$$
$$=\underline{R}^{E0}_{var_m}-\underline{R}^{E0}_{var_{m-2}}-2\,\underline{V}^{E0}_{var_{m-2}}\,T_m-A_{m-1}\left(\underline{V}^{E0}_{var_m}-\underline{V}^{E0}_{var_{m-2}}\right)T_m$$
$$+\left[\left(\frac{A_{m-1}}{2}-\frac{1}{6}I\right)\underline{g}^{E0}_{var_m}+\left(A_{m-1}-I\right)\underline{g}^{E0}_{var_{m-1}}+\left(\frac{A_{m-1}}{2}-\frac{5}{6}I\right)\underline{g}^{E0}_{var_{m-2}}\right]T_m^2 \tag{25}$$

Defining particular groupings in (25) as

$$B_{den_{m-1}}\equiv\left[C^{E0}_{Bvar_{m-2}}\left(G_{Rvar_{m-1}}+G_{Vvar_{m-1}}\right)-A_{m-1}\,C^{E0}_{Bvar_{m-2}}\,G_{Vvar_{m-1}}\right]T_m$$
$$B_{num_{m-1}}\equiv\underline{R}^{E0}_{var_m}-\underline{R}^{E0}_{var_{m-2}}-2\,\underline{V}^{E0}_{var_{m-2}}\,T_m-A_{m-1}\left(\underline{V}^{E0}_{var_m}-\underline{V}^{E0}_{var_{m-2}}\right)T_m$$
$$+\left[\left(\frac{A_{m-1}}{2}-\frac{1}{6}I\right)\underline{g}^{E0}_{var_m}+\left(A_{m-1}-I\right)\underline{g}^{E0}_{var_{m-1}}+\left(\frac{A_{m-1}}{2}-\frac{5}{6}I\right)\underline{g}^{E0}_{var_{m-2}}\right]T_m^2 \tag{26}$$

allows (25) to be solved for $\Delta\underline{\upsilon}^{Bvar}_{var_{m-1}}$:

$$\Delta\underline{\upsilon}^{Bvar}_{var_{m-1}}=B^{-1}_{den_{m-1}}\,B_{num_{m-1}} \tag{27}$$

The same process is used to obtain $\Delta\underline{\upsilon}^{Bvar}_{var_m}$ by returning to (16) and (19), but first solving each for $\Delta\underline{\upsilon}^{Bvar}_{var_{m-1}}\,T_m$. Following the methodology leading to (23), (26), and (27), the equivalent solution for $\Delta\underline{\upsilon}^{Bvar}_{var_m}$ is given by

$$A_m\equiv C^{E0}_{Bvar_{m-2}}\left(G_{Rvar_{m-1}}+G_{Vvar_{m-1}}\right)\left(C^{E0}_{Bvar_{m-2}}\,G_{Vvar_{m-1}}\right)^{-1} \tag{28}$$

$$B_{den_m}\equiv\left(C^{E0}_{Bvar_{m-1}}\,G_{Rvar_m}-A_m\,C^{E0}_{Bvar_{m-1}}\,G_{Vvar_m}\right)T_m$$
$$B_{num_m}\equiv\underline{R}^{E0}_{var_m}-\underline{R}^{E0}_{var_{m-2}}-2\,\underline{V}^{E0}_{var_{m-2}}\,T_m-A_m\left(\underline{V}^{E0}_{var_m}-\underline{V}^{E0}_{var_{m-2}}\right)T_m$$
$$+\left[\left(\frac{A_m}{2}-\frac{1}{6}I\right)\underline{g}^{E0}_{var_m}+\left(A_m-I\right)\underline{g}^{E0}_{var_{m-1}}+\left(\frac{A_m}{2}-\frac{5}{6}I\right)\underline{g}^{E0}_{var_{m-2}}\right]T_m^2 \tag{29}$$

$$\Delta\underline{\upsilon}^{Bvar}_{var_m}=B^{-1}_{den_m}\,B_{num_m} \tag{30}$$

Eq. (27) for $\Delta\underline{\upsilon}^{Bvar}_{var_{m-1}}$ with (23) and (26), and (30) for $\Delta\underline{\upsilon}^{Bvar}_{var_m}$ with (28) and (29), are the integrated specific force increments for Method 2, both functions of variation trajectory position

$\underline{R}_{var}^{E0}$ and velocity $\underline{V}_{var}^{E0}$ at the end of cycles *m* and *m*-2. It is important to understand that these equations must be processed as a set, once every other *m* cycle using $\underline{R}_{var_m}^{E0}$, $\underline{R}_{var_{m-2}}^{E0}$, $\underline{V}_{var_m}^{E0}$, $\underline{V}_{var_{m-2}}^{E0}$ values prescribed by position/velocity constraints (6) – (7) and (10). This assures that the basis for (27) and (30), the specified position/velocity constraints will both be satisfied at *m*-2 and *m*, when the computed $\Delta\underline{\upsilon}_{var_{m-1}}^{Bvar}$, $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ integrated specific force increments are applied sequentially (at time interval *m*-1 followed by *m*). To clarify this point, it is expeditious to define another cycle *n* representing the occurrence of even *m* cycles, i.e., at $m = -2$, 0, +2, etc. With this definition we can write that at even *m* cycles, $m = 2n$, and at odd *m* cycles, $m = 2n-1$. Using this representation, (23), (26) - (27), (28), and (29) – (30) then summarize as follows:

$$
\begin{aligned}
A_{2n-1} &\equiv C_{Bvar_{2n-1}}^{E0}\, G_{Rvar_{2n}} \left(C_{Bvar_{2n-1}}^{E0}\, G_{Vvar_{2n}}\right)^{-1} \\
B_{den_{2n-1}} &\equiv \left[C_{Bvar_{2n-2}}^{E0}\left(G_{Rvar_{2n-1}} + G_{Vvar_{2n-1}}\right) - A_{2n-1}\, C_{Bvar_{2n-2}}^{E0}\, G_{Vvar_{2n-1}} \right] T_m \\
B_{num_{2n-1}} &\equiv \underline{R}_{var_{2n}}^{E0} - \underline{R}_{var_{2n-2}}^{E0} - 2\,\underline{V}_{var_{2n-2}}^{E0}\, T_m - A_{2n-1}\left(\underline{V}_{var_{2n}}^{E0} - \underline{V}_{var_{2n-2}}^{E0}\right) T_m \\
&\quad + \left[\left(\frac{A_{2n-1}}{2} - \frac{1}{6} I\right) \underline{g}_{var_{2n}}^{E0} + \left(A_{2n-1} - I\right) \underline{g}_{var_{2n-1}}^{E0} + \left(\frac{A_{2n-1}}{2} - \frac{5}{6} I\right) \underline{g}_{var_{2n-2}}^{E0} \right] T_m^2 \\
\Delta\underline{\upsilon}_{var_{2n-1}}^{Bvar} &= B_{den_{2n-1}}^{-1}\, B_{num_{2n-1}} \\
A_{2n} &\equiv C_{Bvar_{2n-2}}^{E0}\left(G_{Rvar_{2n-1}} + G_{Vvar_{2n-1}}\right)\left(C_{Bvar_{2n-2}}^{E0}\, G_{Vvar_{2n-1}}\right)^{-1} \\
B_{den_{2n}} &\equiv \left(C_{Bvar_{2n-1}}^{E0}\, G_{Rvar_{2n}} - A_{2n}\, C_{Bvar_{2n-1}}^{E0}\, G_{Vvar_{2n}}\right) T_m \\
B_{num_{2n}} &\equiv \underline{R}_{var_{2n}}^{E0} - \underline{R}_{var_{2n-2}}^{E0} - 2\,\underline{V}_{var_{2n-2}}^{E0}\, T_m - A_{2n}\left(\underline{V}_{var_{2n}}^{E0} - \underline{V}_{var_{2n-2}}^{E0}\right) T_m \\
&\quad + \left[\left(\frac{A_{2n}}{2} - \frac{1}{6} I\right) \underline{g}_{var_{2n}}^{E0} + \left(A_{2n} - I\right) \underline{g}_{var_{2n-1}}^{E0} + \left(\frac{A_{2n}}{2} - \frac{5}{6} I\right) \underline{g}_{var_{2n-2}}^{E0} \right] T_m^2 \\
\Delta\underline{\upsilon}_{var_{2n}}^{Bvar} &= B_{den_{2n}}^{-1}\, B_{num_{2n}}
\end{aligned}
\tag{31}
$$

Importantly, each **subscript** in (31) identifies the ***m* cycle** time instant value for that parameter.

Execution of (31) requires C_{Bvar}^{E0}, G_{Vvar}, G_{Rvar}, $\underline{g}_{var}^{E0}$ at each *m* cycle, and $\underline{R}_{var}^{E0}$, $\underline{V}_{var}^{E0}$ at each *n* cycle. The G_{Vvar}, G_{Rvar} parameters are computed with (3) and (4) using $\Delta\underline{\alpha}_{var_m}^{Bvar}$ from (8). Eq. (8) for $C_{Bvar_m}^{E0}$ is a function of the previous cycle transposed value of $C_{Bvar_m}^{E0}$,

$C^{E0}_{Bref_m}$ from reference trajectory attitude, and $C^{Bref}_{Bvar_m}$, a user specified variation in the $Bvar$ frame attitude relative to the $Bref$ frame. The $\underline{R}^{E0}_{var}$, $\underline{V}^{E0}_{var}$ parameters are functions of $\underline{S}^{E0}$, $\underline{R}^{E0}_{ref}$, $\underline{V}^{E0}_{ref}$ in (6) and (10); the *n* cycle values of $\underline{S}^{E0}$ are used for $\underline{R}^{E0}_{var}$, the *m* cycle values are used for $\underline{V}^{E0}_{var}$. The $\underline{S}^{E0}$ parameter is a function of user specified $\underline{l}^{Bref_m}$ and $\underline{s}^{Bvar_m}$ lever arm data (and $C^{Bref}_{Bvar_m}$) as calculated with (7). The $\underline{V}^{E0}_{ref}$ parameter is obtained from the reference trajectory, either directly, or as described (for example) in Appendix E. The $\underline{R}^{E0}_{ref}$ position parameter is obtained from the reference trajectory, or from Appendix E as described in the second paragraph following (9). The $\underline{g}^{E0}_{var}$ parameter is a function of $\underline{R}^{E0}_{var}$ calculated with (6) – (7) from $\underline{R}^{E0}_{ref}$ and $\underline{S}^{E0}$. Appendices A and B show how $\underline{g}^{E0}_{var}$ can be determined from $\underline{R}^{E0}_{var}$, either directly, or as a first order expansion around $\underline{R}^{E0}_{ref}$.

GENERATING VARIATION AND REFERENCE TRAJECTORY OUTPUTS

Once $\Delta\underline{\alpha}^{Bvar}_{var_m}$ and $\Delta\underline{\upsilon}^{Bvar}_{var_m}$ have been determined, they can be processed through an appropriate set of integration algorithms to determine attitude/velocity/position navigation data for output, e.g., (2), (3), and (4) if outputs are to be provided relative to non-rotating inertial E_0 coordinates, or the equivalent for a wander azimuth output format as described in Appendix E. Outputs would include $\Delta\underline{\alpha}^{Bvar}_{var_m}$ and $\Delta\underline{\upsilon}^{Bvar}_{var_m}$ that generated the navigation data.

The basic motivation for developing a variation trajectory is for use in Kalman aided INS simulations. Then, variation trajectory data would represent the “true” navigation state at the aided INS location with the angular-rate/specific-force representing the aided INS gyro/accelerometer inputs. To complete the simulation, reference trajectory data would also be provided as a simulated source of master navigation data for Kalman aiding filter input (for comparison with the aided INS output in the Kalman filter “measurement”). Representative error sources would be added to the reference trajectory navigation data to simulate the master reference output. Aided INS gyro/accelerometer outputs for navigation data generation would be created by adding representative errors to the variation trajectory angular-rate/specific-force. Aided INS outputs would be generated by processing the simulated gyro/accelerometer data through the anticipated aided INS navigation algorithms. Aided INS performance would then be evaluated by comparing the aided INS output navigation data with the equivalent variation trajectory data.

It is important to recognize that for the previous process, the computation routines used to generate the variation trajectory outputs must be the same as those used in creating the reference trajectory. This is the only way to assure that the difference between the reference and variation trajectories will accurately reflect attitude/velocity/position constraints (6), (7), (8), and (10) (for Method 2). For situations where the reference trajectory used in (31) for Method 2 (to generate variation trajectory angular-rate/specific-force) is provided directly (rather than by parallel integration), an equivalent reference must be generated, created by an equivalent set of reference angular-rate/specific-force operating through the same set of integration algorithms used for variation trajectory generation. The reference angular-rate/specific-force would be created from the original reference navigation data using (31), but with original reference position $\underline{R}_{ref_m}^{E0}$ replacing $\underline{R}_{var_m}^{E0}$ at alternate *m* cycles (i.e., the equivalent of $\underline{R}_{var_m}^{E0}$ with zero $\underline{S}^{E0}$ offset), original reference attitude for $C_{Bvar_m}^{E0}$ (i.e., $C_{Bvar_m}^{E0}$ from (8) with identity $C_{Bvar_m}^{Bref_m}$ attitude offset) at each *m* cycle, and velocity from (10) (using $\underline{R}_{ref_m}^{E0}$ for $\underline{R}_{var_m}^{E0}$).

TEST EXAMPLE FOR EVALUATING METHOD 1 VERSUS METHOD 2 PERFORMANCE

This section describes a test example used to analytically evaluate Method 1 and Method 2 specific force algorithm performance and its impact on velocity/position generated with (3) and (4). The section derives theoretical analytical specific-force/velocity/position data for the test example for later comparison with equivalent Method 1 and 2 generated results.

TEST EXAMPLE DESCRIPTION

For comparing Method 1 versus Method 2 performance, consider a test example having $C_{Bvar_m}^{Bref_m} = I$ so that from (5), $C_{Bref_m}^{E0} = C_{Bvar_m}^{E0}$, and $\underline{\omega}_{ref}$, the angular rate of the $Bref$ frame relative to non-rotating E_0 inertial space then represents $\underline{\omega}_{var}$, the angular rate of the variation trajectory relative to inertial space. Also assume for the example that $\underline{\omega}_{var}$ is zero prior to time instant $m = -9$, constant from $m = -9$ to $m = 0$ (call it $\underline{\omega}$), and another constant for $m > 0$ (call it $\underline{\omega}'$).

Under the variation trajectory structure, the previously defined $\underline{\omega}_{var}$ example history (of zero, constant $\underline{\omega}$, and constant $\underline{\omega}'$ angular rates) would be represented in $Bvar$ coordinates by integrated angular rate increment sequences of zero for $m < -9$, constant $\Delta\underline{\alpha}_{var}^{Bvar} = \underline{\omega}^{Bvar}\, T_m$ for $0 \geq m > -9$, and another constant $\Delta\underline{\alpha}'^{Bvar}_{var} = \underline{\omega}'^{Bvar}\, T_m$ for $m > 0$. Based on these settings, the following equivalent to (2) – (4) will be used for C_{E0}^{Bvar}, G_{Vvar_m}, and G_{Rvar_m} determination under the example conditions:

$$\text{For } m \le -9: \quad C_{Bvar_m}^{E0} = C_{Bvar_{-9}}^{E0}$$

$$\text{For } 0 \ge m > -9:$$

$$C_{Bvar_m}^{E0} \approx C_{Bvar_{m-1}}^{E0} \left[I + \left(\Delta \underline{\alpha}_{var}^{Bvar} \times \right) + \frac{1}{2} \left(\Delta \underline{\alpha}_{var}^{Bvar} \times \right)^2 \right] \approx C_{Bvar_{m-1}}^{E0} \left[I + \left(\Delta \underline{\alpha}_{var}^{Bvar} \times \right) \right] \tag{32}$$

$$\text{For } m > 0:$$

$$C_{Bvar_m}^{E0} \approx C_{Bvar_{m-1}}^{E0} \left[I + \left(\Delta \underline{\alpha}'^{Bvar}_{var} \times \right) + \frac{1}{2} \left(\Delta \underline{\alpha}'^{Bvar}_{var} \times \right)^2 \right] \approx C_{Bvar_{m-1}}^{E0} \left[I + \left(\Delta \underline{\alpha}'^{Bvar}_{var} \times \right) \right]$$

$$\text{For } m \le -9: \quad G_{Vvar_m} = I \quad G_{Rvar_m} = \frac{1}{2} I$$

$$\text{For } 0 \ge m > -9: \quad G_{Vvar_m} \approx I + \frac{1}{2} \left(\Delta \underline{\alpha}_{var}^{Bvar} \times \right) \quad G_{Rvar_m} \approx \frac{1}{2} \left[I + \frac{1}{3} \left(\Delta \underline{\alpha}_{var}^{Bvar} \times \right) \right] \tag{33}$$

$$\text{For } m > 0: \quad G_{Vvar_m} \approx I + \frac{1}{2} \left(\Delta \underline{\alpha}'^{Bvar}_{var} \times \right) \quad G_{Rvar_m} \approx \frac{1}{2} \left[I + \frac{1}{3} \left(\Delta \underline{\alpha}'^{Bvar}_{var} \times \right) \right]$$

To evaluate $\underline{R}_{var_m}^{E0}$ in (6) for position constraint application requires specifying the form of $\underline{R}_{ref_m}^{E0}$ and the associated $\underline{S}_m^{E0}$ parameters in (7). For the test example we specify in (7) that $C_{Bvar_m}^{Bref} = I$, $\underline{s}^{Bvar_m} = 0$, and $\underline{l}^{Bref_m}$ be constant (call it $\underline{l}^{Bref}$), hence, (7) simplifies to:

$$\underline{S}_m^{E0} = C_{Bvar_m}^{E0} \underline{l}^{Bref} \tag{34}$$

Reference position $\underline{R}_{ref_m}^{E0}$ in (6) is created by $\underline{V}_{ref_m}^{E0}$ velocity which, for the test example, is set to the constant $\underline{V}_{ref_0}^{E0}$. Additionally, $\underline{R}_{ref_m}^{E0}$ position is specified to be zero at $m = 0$. Then $\underline{R}_{ref_m}^{E0}$ in general becomes

$$\underline{R}_{ref}^{E0} = \underline{V}_{ref_0}^{E0} \left(t - t_0 \right) \tag{35}$$

where t_0 is general time *t* at *m* cycle time instant 0. In terms of variation trajectory fixed T_m time interval *m* cycles, (35) is equivalently:

$$\underline{R}_{ref_m}^{E0} = m \, \underline{V}_{ref_0}^{E0} \, T_m \tag{36}$$

With (34) and (36), position constraint (6) becomes:

$$\underline{R}_{var_m}^{E0} = m\,\underline{V}_{ref_0}^{E0}\;T_m + C_{Bvar_m}^{E0}\,\underline{l}^{Bref} \tag{37}$$

Finally, for the test example it is assumed that $\underline{V}_{ref_0}^{E0}$ is slow enough and $\underline{l}^{Bref}$ relatively small enough that $\underline{R}_{var_m}^{E0}$ in (37) has only a small change over the duration of the test example time period. Thus, gravity in (3) and (4) (a function of $\underline{R}_{var_m}^{E0}$) can be approximated as constant, i.e.,

$$\underline{g}_{var_m}^{E0} \approx \text{Constant} \equiv \underline{g}_{avg}^{E0} \tag{38}$$

THEORETICAL SOLUTION UNDER TEST EXAMPLE CONDITIONS

On a continuous basis, (6) with (34) – (35) becomes:

$$\underline{R}_{var}^{E0} = \underline{V}_{ref_0}^{E0}\left(t - t_0\right) + C_{Bvar}^{E0}\,\underline{l}^{Bref} \tag{39}$$

With $\dot{C}_{Bvar}^{E0} = C_{Bvar}^{E0}\left(\underline{\omega}_{var}^{Bvar}\times\right)$ from [1, Eq. (3.3.2-6)], the derivative of (39) finds

$$\begin{aligned}\dot{\underline{R}}_{var}^{E0} \equiv \underline{V}_{var}^{E0} = \underline{V}_{ref_0}^{E0} + \dot{C}_{Bvar}^{E0}\,\underline{l}^{Bref} &= \underline{V}_{ref_0}^{E0} + C_{Bvar}^{E0}\left(\underline{\omega}_{var}^{Bvar}\times\right)\underline{l}^{Bref} \\ &= \underline{V}_{ref_0}^{E0} + C_{Bvar}^{E0}\left(\underline{\omega}_{var}^{Bvar}\times\underline{l}^{Bref}\right)\end{aligned} \tag{40}$$

During the test example time segments of constant angular rate $\underline{\omega}_{var}^{Bvar}$, the derivative of (40) with $\dot{C}_{Bvar}^{E0} = C_{Bvar}^{E0}\left(\underline{\omega}_{var}^{Bvar}\times\right)$ obtains $\underline{a}_{var}^{E0}$, the acceleration of $\underline{R}_{var}^{E0}$:

$$\ddot{\underline{R}}_{var}^{E0} \equiv \underline{a}_{var}^{E0} = \dot{C}_{Bvar}^{E0}\left(\underline{\omega}_{var}^{Bvar}\times\underline{l}^{Bref}\right) = C_{Bvar}^{E0}\left[\underline{\omega}_{var}^{Bvar}\times\left(\underline{\omega}_{var}^{Bvar}\times\underline{l}^{Bref}\right)\right] \tag{41}$$

Equating $\underline{a}_{var}^{E0}$ acceleration in (41) to specific force $\underline{a}_{SF}^{E0}$ plus gravity, and from (38), approximating gravity for the example as constant $\underline{g}_{avg}^{E0}$, (41) becomes for $\underline{a}_{SF}^{E0}$:

$$\underline{a}_{SF}^{E0} = C_{Bvar}^{E0}\left[\underline{\omega}_{var}^{Bvar}\times\left(\underline{\omega}_{var}^{Bvar}\times\underline{l}^{Bref}\right)\right] + \underline{g}_{avg}^{E0} \tag{42}$$

Transforming (42) to the *Bvar* frame then obtains the resulting $\underline{a}_{SF}^{Bvar}$ specific force (the acceleration that would be measured by accelerometers aligned with *Bvar* coordinates):

$$\underline{a}_{SF}^{Bvar} = \underline{\omega}_{var}^{Bvar} \times \left(\underline{\omega}_{var}^{Bvar} \times \underline{l}^{Bref} \right) - C_{E0}^{Bvar} \; \underline{g}_{avg}^{E0} \tag{43}$$

For the assumed $\underline{\omega}_{var}^{Bvar}$ values defined at the start of this section, (39), (40), and (43) then translate into the following theoretical solution for the test example:

$$\begin{gathered}
\text{For } t < t_{m=-9} \\
\underline{\omega}_{var}^{Bvar} = 0 \quad \underline{a}_{SF}^{Bvar} = - C_{E0}^{Bvar} \; \underline{g}_{avg}^{E0} \quad \underline{V}_{var}^{E0} = \underline{V}_{ref_0}^{E0} \quad \underline{R}_{var}^{E0} = \underline{V}_{ref_0}^{E0} \left(t - t_0 \right) + C_{Bvar_{m=-9}}^{E0} \, \underline{l}^{Bref} \\
\text{For } t_{m=-9} < t < t_0 : \\
\underline{\omega}_{var}^{Bvar} = \underline{\omega}^{Bvar} \qquad \underline{a}_{SF}^{Bvar} = \underline{\omega}^{Bvar} \times \left(\underline{\omega}^{Bvar} \times \underline{l}^{Bref} \right) - C_{E0}^{Bvar} \; \underline{g}_{avg}^{E0} \\
\underline{V}_{var}^{E0} = \underline{V}_{ref_0}^{E0} + C_{Bvar}^{E0} \left(\underline{\omega}^{Bvar} \times \underline{l}^{Bref} \right) \qquad \underline{R}_{var}^{E0} = \underline{V}_{ref_0}^{E0} \left(t - t_0 \right) + C_{Bvar}^{E0} \, \underline{l}^{Bref} \\
\text{For } t > t_0 : \\
\underline{\omega}_{var}^{Bvar} = \underline{\omega}'^{Bvar} \qquad \underline{a}_{SF}^{Bvar} = \underline{\omega}'^{Bvar} \times \left(\underline{\omega}'^{Bvar} \times \underline{l}^{Bref} \right) - C_{E0}^{Bvar} \; \underline{g}_{avg}^{E0} \\
\underline{V}_{var}^{E0} = \underline{V}_{ref_0}^{E0} + C_{Bvar}^{E0} \left(\underline{\omega}'^{Bvar} \times \underline{l}^{Bref} \right) \qquad \underline{R}_{var}^{E0} = \underline{V}_{ref_0}^{E0} \left(t - t_0 \right) + C_{Bvar}^{E0} \, \underline{l}^{Bref}
\end{gathered} \tag{44}$$

where $t_{m=-9}$ is time t at the instantaneous ending of m cycle -9, and t_0 is time t at the instantaneous ending of m cycle 0.

Note the lack of specificity in (44) for specific force $\underline{a}_{SF}^{Bvar}$ at the $t = t_{m=-9}$ and $t = t_0$ time instants. At these times, the $\underline{\omega}_{var}^{Bvar}$ angular rate changes instantaneously from zero to $\underline{\omega}^{Bvar}$ and from $\underline{\omega}^{Bvar}$ to $\underline{\omega}'^{Bvar}$. This is accompanied by a change in $\underline{V}_{var}^{E0}$ velocity of $C_{Bvar}^{E0} \left(\underline{\omega}^{Bvar} \times \underline{l}^{Bref} \right)$ at $t = t_{m=-9}$, followed by a $\underline{V}_{var}^{E0}$ change of $C_{Bvar}^{E0} \left(\underline{\omega}'^{Bvar} \times \underline{l}^{Bref} \right)$ minus $C_{Bvar}^{E0} \left(\underline{\omega}^{Bvar} \times \underline{l}^{Bref} \right)$ at $t = t_0$. For theoretical representation in (44), the instantaneous effects can be created by specific force impulses at these time points (i.e., of infinite amplitude and zero time width), clearly an unrealistic effect for simulation (or reality).

Under the variation trajectory simulation structure, $\underline{\omega}_{var}^{Bvar}$ values of zero, constant $\underline{\omega}^{Bvar}$, and constant $\underline{\omega}'^{Bvar}$ angular rates in (44) are represented by sequences of finite amplitude integrated angular rate increments, each increment over the same T_m fixed time interval (i.e., of zero for $m < -9$, constant $\Delta \underline{\alpha}_{var}^{Bvar} = \underline{\omega}^{Bvar} \, T_m$ for $0 \ge m > -9$, and constant $\Delta \underline{\alpha}'^{Bvar}_{var} = \underline{\omega}'^{Bvar} \, T_m$ for $m > 0$). Similarly, the $\underline{a}_{SF}^{Bvar}$ specific force vector in (44) would be represented by its equivalent finite integrated specific force increment $\Delta \underline{\upsilon}_{var}^{Bvar} = \underline{a}_{SF}^{Bvar} \, T_m$ over each m cycle. Thus, the equivalent to (44) for the variation trajectory would be:

$$
\begin{gathered}
\text{For } m \leq -9: \\
\underline{\omega}_{var}^{Bvar} T_m = 0 \qquad \Delta\underline{\upsilon}_{var_m}^{Bvar} = -C_{E0_{-9}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \qquad \underline{V}_{var_m}^{E0} = \underline{V}_{ref_0}^{E0} \\
\underline{R}_{var_m}^{E0} = m\, \underline{V}_{ref_0}^{E0}\, T_m + C_{Bvar_{-9}}^{E0}\, \underline{l}^{Bref} \\
\text{For } -9 < m \leq 0: \\
\underline{\omega}_{var}^{Bvar} T_m = \Delta\underline{\alpha}_{var}^{Bvar} \qquad \Delta\underline{\upsilon}_{var_m}^{Bvar} \approx \Delta\underline{\alpha}_{var}^{Bvar} \times \left(\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m} \right) - C_{E0_{m-1/2}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \\
\underline{V}_{var_m}^{E0} \approx \underline{V}_{ref_0}^{E0} + C_{Bvar_{m-1/2}}^{E0} \left(\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m} \right) \qquad \underline{R}_{var_m}^{E0} = m\, \underline{V}_{ref_0}^{E0}\, T_m + C_{Bvar_m}^{E0}\, \underline{l}^{Bref} \\
\text{For } m > 0: \\
\underline{\omega}_{var}^{Bvar} T_m = \Delta\underline{\alpha}'^{Bvar}_{var} \qquad \Delta\underline{\upsilon}_{var_m}^{Bvar} \approx \Delta\underline{\alpha}'^{Bvar}_{var} \times \left(\Delta\underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m} \right) - C_{E0_{m-1/2}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \\
\underline{V}_{var}^{E0} \approx \underline{V}_{ref_0}^{E0} + C_{Bvar_{m-1/2}}^{E0} \left(\Delta\underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m} \right) \qquad \underline{R}_{var_m}^{E0} = m\, \underline{V}_{ref_0}^{E0}\, T_m + C_{Bvar_m}^{E0}\, \underline{l}^{Bref}
\end{gathered}
\tag{45}
$$

where $C_{E0_{m-1/2}}^{Bvar}$, $C_{Bvar_{m-1/2}}^{E0}$ are values for C_{E0}^{Bvar}, C_{Bvar}^{E0} halfway between $C_{E0_{m-1}}^{Bvar}$, $C_{Bvar_{m-1}}^{E0}$ and $C_{E0_m}^{Bvar}$, $C_{Bvar_m}^{E0}$.

The problematic effects manifested in (44) at $t = t_{m=-9}$ and $t = t_0$ remain in the (45) equivalent representation: the unaccounted for change in $\underline{V}_{var_m}^{E0}$ velocity at the $m = -9$ and $m = 0$ cycle times. The problem was resolved in (44) by employing $\underline{a}_{SF}^{Bvar}$ impulses to instantaneously transition velocity $\underline{V}_{var_m}^{E0}$ to the proper values without impacting $\underline{R}_{var}^{E0}$ position. But with the assumed variation trajectory generator structure, impulsive velocity correction in (45) is not a realistic option because specific force is modeled as a finite constant over each $m-1$ to m time period, each time period being of the same T_m time duration. As a result, achieving the required (45) velocity changes can only be accomplished in the trajectory generator by modifying the $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ increment finite amplitudes. The realism of the resulting modified $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ profile and its impact on $\underline{V}_{var_m}^{E0}$, $\underline{R}_{var_m}^{E0}$ velocity/position is the key distinction between the Method 1 approach for $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ based on (9), and the Method 2 approach for $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ based on (31). Appendices H and I in [2] analytically derive the Method 1 and Method 2 generated performance under the test example conditions. Results are presented and analyzed next.

METHOD 1 PERFORMANCE UNDER THE TEST EXAMPLE

The following subsections present and discuss performance results analytically derived in [2, Appendix G] for $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ integrated specific force increments generated under test example input conditions using the Method 1 design approach: (9) for $\Delta\underline{\upsilon}_{var_m}^{Bvar}$, with substitution in (3) - (4) for the $\underline{V}_{var_m}^{E0}$, $\underline{R}_{var_m}^{E0}$ velocity/position response.

METHOD 1 SPECIFIC FORCE

The specific force profile generated by (9) with Method 1 is analytically derived in [2, Appendix G] leading to (G-14), then renumbered and shown next.

$$\text{For } m < -8:$$
$$\Delta\underline{\upsilon}_{var_m}^{Bvar} = -\,C_{E0_{-9}}^{Bvar}\,\underline{g}_{avg}^{E0}\,T_m$$
$$\text{For } m = -8, -6, -4, -2, 0:$$
$$\Delta\underline{\upsilon}_{var_m}^{Bvar} \approx 2\,\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m} - \left[I - \frac{1}{2}\left(\Delta\underline{\alpha}_{var}^{Bvar} \times\right)\right]\left(C_{E0_{m-1}}^{Bvar}\,\underline{g}_{avg}^{E0}\,T_m\right)$$
$$-\frac{1}{6}\Delta\underline{\alpha}_{var}^{Bvar} \times \left(C_{E0_{m-1}}^{Bvar}\,\underline{g}_{avg}^{E0}\,T_m\right)$$
$$\text{For } m = -7, -5, -3, -1:$$
$$\Delta\underline{\upsilon}_{var_m}^{Bvar} = -2\,\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m} - \left[I - \frac{1}{2}\left(\Delta\underline{\alpha}_{var}^{Bvar} \times\right)\right]\left(C_{E0_{m-1}}^{Bvar}\,\underline{g}^{E0}\,T_m\right)$$
$$+\frac{1}{6}\Delta\underline{\alpha}_{var}^{Bvar} \times \left(C_{E0_{m-1}}^{Bvar}\,\underline{g}_{avg}^{E0}\,T_m\right)$$
$$\text{For } m = 1, 3, 5, \cdots: \tag{46}$$
$$\Delta\underline{\upsilon}_{var_m}^{Bvar} = -\,2\,\Delta\underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m} - \left[I - \frac{1}{2}\left(\Delta\underline{\alpha}'^{Bvar}_{var} \times\right)\right]\left(C_{E0_{m-1}}^{Bvar}\,\underline{g}_{avg}^{E0}\,T_m\right)$$
$$+\frac{1}{6}\Delta\underline{\alpha}'^{Bvar}_{var} \times \left(C_{E0_{m-1}}^{Bvar}\,\underline{g}_{avg}^{E0}\,T_m\right)$$
$$+\,4\left(\Delta\underline{\alpha}'^{Bvar}_{var} - \Delta\underline{\alpha}_{var}^{Bvar}\right) \times \frac{\underline{l}^{Bref}}{T_m} - \frac{1}{3}\left(\Delta\underline{\alpha}'^{Bvar}_{var} - \Delta\underline{\alpha}_{var}^{Bvar}\right) \times \left(C_{E0_{m-1}}^{Bvar}\,\underline{g}_{avg}^{E0}\,T_m\right)$$
$$\text{For } m = 2, 4, 6, \cdots:$$
$$\Delta\underline{\upsilon}_{var_m}^{Bvar} = 2\,\Delta\underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m} - \left[I - \frac{1}{2}\left(\Delta\underline{\alpha}'^{Bvar}_{var}\right) \times\right]\left(C_{E0_{m-1}}^{Bvar}\,\underline{g}_{avg}^{E0}\,T_m\right)$$
$$-\frac{1}{6}\Delta\underline{\alpha}'^{Bvar}_{var} \times \left(C_{E0_{m-1}}^{Bvar}\,\underline{g}_{avg}^{E0}\,T_m\right)$$
$$-\,4\left(\Delta\underline{\alpha}'^{Bvar}_{var} - \Delta\underline{\alpha}_{var}^{Bvar}\right) \times \frac{\underline{l}^{Bref}}{T_m} + \frac{1}{3}\left(\Delta\underline{\alpha}'^{Bvar}_{var} - \Delta\underline{\alpha}_{var}^{Bvar}\right) \times \left(C_{E0_{m-1}}^{Bvar}\,\underline{g}_{avg}^{E0}\,T_m\right)$$

Based on the theoretical solution in (45), it might be expected that under rotation, each (46) integrated specific force $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ increment would contain a component $C_{E0_{m-1/2}}^{Bvar}\,\underline{g}_{avg}^{E0}\,T_m$ to balance gravity. Eqs. (46) show that such a component is indeed present, as represented by the equivalent $\left[I - \frac{1}{2}\left(\Delta\underline{\alpha}_{var}^{Bvar} \times\right)\right]\left(C_{E0_{m-1}}^{Bvar}\,\underline{g}_{avg}^{E0}\,T_m\right)$ for $m \leq 0$ and by $\left[I - \frac{1}{2}\left(\Delta\underline{\alpha}'^{Bvar}_{var} \times\right)\right]\left(C_{E0_{m-1}}^{Bvar}\,\underline{g}_{avg}^{E0}\,T_m\right)$ for $m > 0$. However, (46) differs from (45) by the addition of other gravity terms, and in the form of the lever arm ($\underline{l}^{Bref}$) terms.

Method 1 Specific force Response To Gravity Effects

In addition to the equivalent $C_{E0_{m-1/2}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m$ gravity term in (45) discussed previously, (46) includes a $\frac{1}{6}\Delta\underline{\alpha}_{var}^{Bvar} \times \left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$ term triggered at *m* cycle -9 (when $\underline{\omega}_{var}^{Bvar}$ changes from 0 to $\underline{\omega}^{Bvar}$) that oscillates between *m* cycles (plus for odd *m* cycles, minus for even *m* cycles) until *m* cycle 0. Similarly, a cyclic $\frac{1}{6}\Delta\underline{\alpha}'^{Bvar}_{var} \times \left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$ gravity term is triggered at *m* cycle 0 (when $\underline{\omega}_{var}^{Bvar}$ changes from $\underline{\omega}^{Bvar}$ to $\underline{\omega}'^{Bvar}$) that persists thereafter. In addition, a second oscillatory gravity term $\frac{1}{3}\left(\Delta\underline{\alpha}'^{Bvar}_{var} - \Delta\underline{\alpha}_{var}^{Bvar} \right) \times \left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$ is triggered by the *m* cycle 0 angular rate transition (from $\underline{\omega}^{Bvar}$ to $\underline{\omega}'^{Bvar}$) that persists thereafter. It is informative to note how the added gravity terms interact after *m* = 0 under two conditions, when $\Delta\underline{\alpha}'^{Bvar}_{var} = \Delta\underline{\alpha}_{var}^{Bvar}$ and when $\Delta\underline{\alpha}'^{Bvar}_{var} = 0$.

When $\Delta\underline{\alpha}'^{Bvar}_{var} = \Delta\underline{\alpha}_{var}^{Bvar}$, the $\frac{1}{3}\left(\Delta\underline{\alpha}'^{Bvar}_{var} - \Delta\underline{\alpha}_{var}^{Bvar} \right) \times \left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$ term vanishes, the $\frac{1}{6}\Delta\underline{\alpha}'^{Bvar}_{var} \times \left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$ term becomes $\frac{1}{6}\Delta\underline{\alpha}_{var}^{Bvar} \times \left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$, and the $\left[I - \frac{1}{2}\left(\Delta\underline{\alpha}'^{Bvar}_{var} \times \right) \right]\left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$ term goes to $\left[I - \frac{1}{2}\left(\Delta\underline{\alpha}_{var}^{Bvar} \times \right) \right]\left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$, i.e., a continuation of the solution that was operating prior to *m* = 0. When $\Delta\underline{\alpha}'^{Bvar}_{var} = 0$, the $\frac{1}{6}\Delta\underline{\alpha}'^{Bvar}_{var} \times \left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$ term vanishes, the $-\left[I - \frac{1}{2}\left(\Delta\underline{\alpha}'^{Bvar}_{var} \right) \times \right]\left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$ term goes to $-\left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$ as in the theoretical (45) solution for zero $\Delta\underline{\alpha}'^{Bvar}_{var}$, and the $\frac{1}{3}\left(\Delta\underline{\alpha}'^{Bvar}_{var} - \Delta\underline{\alpha}_{var}^{Bvar} \right) \times \left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$ term becomes $-\frac{1}{3}\Delta\underline{\alpha}_{var}^{Bvar} \times \left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$, twice the magnitude of the $\frac{1}{6}\Delta\underline{\alpha}_{var}^{Bvar} \times \left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$ oscillatory term operating prior to *m* = 0. What this represents is the combination of two sustaining $\frac{1}{6}\Delta\underline{\alpha}_{var}^{Bvar} \times \left(C_{E0_{m-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$ oscillations, the one triggered at $m = -9$, and the second triggered at *m* = 0 when $\underline{\omega}_{var}^{Bvar}$ went from $\underline{\omega}^{Bvar}$ to zero (i.e., to a zero $\Delta\underline{\alpha}'^{Bvar}_{var}$).

Method 1 Specific force Response To Lever Arm Effects

For the lever arm ($\underline{l}^{Bref}$) terms, a comparison between the theoretical (45) and Method 1 (46) solutions shows a significant difference. For (45), these terms appear in the centripetal components $\Delta\underline{\alpha}_{var}^{Bvar} \times \left(\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m} \right)$ and $\Delta\underline{\alpha}'^{Bvar}_{var} \times \left(\Delta\underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m} \right)$ that are constant during the m = -9 to 0 and m > 0 time periods. Identical terms would be present in (46), however, due to the first order approximation accuracy in (46), they have been dropped as second order compared to the three more prominent cyclic first order terms that are not present in (45). The first cyclic first order term is $2\,\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m}$ triggered by the change in angular rate at m = -9, the second is $2\,\Delta\underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m}$ triggered by the change in angular rate at m = 0, the third is $4\left(\Delta\underline{\alpha}'^{Bvar}_{var} - \Delta\underline{\alpha}_{var}^{Bvar}\right) \times \frac{\underline{l}^{Bref}}{T_m}$ triggered by the change in angular rate at m = 0. As for the additional cyclic gravity terms discussed previously, it is informative to note how the sustained $\underline{l}^{Bref}$ oscillation terms interact after m = 0 in (46) under two conditions, when $\Delta\underline{\alpha}'^{Bvar}_{var} = \Delta\underline{\alpha}_{var}^{Bvar}$ and when $\Delta\underline{\alpha}'^{Bvar}_{var} = 0$.

When $\Delta\underline{\alpha}'^{Bvar}_{var} = \Delta\underline{\alpha}_{var}^{Bvar}$, the $4\left(\Delta\underline{\alpha}'^{Bvar}_{var} - \Delta\underline{\alpha}_{var}^{Bvar}\right) \times \frac{\underline{l}^{Bref}}{T_m}$ term vanishes, and the $2\,\Delta\underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m}$ term becomes $2\,\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m}$, i.e., a continuation of the $2\,\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m}$ oscillatory term that was operating prior to m = 0. When $\Delta\underline{\alpha}'^{Bvar}_{var} = 0$, the $2\,\Delta\underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m}$ term vanishes, and the $4\left(\Delta\underline{\alpha}'^{Bvar}_{var} - \Delta\underline{\alpha}_{var}^{Bvar}\right) \times \frac{\underline{l}^{Bref}}{T_m}$ term becomes $-\,4\,\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m}$, twice the magnitude of the $2\,\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m}$ oscillatory term operating prior to m = 0. What this represents is the combination of two sustaining $2\,\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m}$ oscillations, the one triggered at $m = -9$, and a second triggered at m = 0.

METHOD 1 VELOCITY

The [2, Appendix G] derived velocity result in (G-6) for Method 1 velocity under test example conditions is renumbered and shown next.

For $m \le -9$:

$$\underline{V}_{var_m}^{E0} = \underline{V}_{ref_0}^{E0}$$

For $m = -8, -6, -4, -2, 0$:

$$\underline{V}_{var_m}^{E0} \approx \underline{V}_{ref_0}^{E0} + 2\, C_{Bvar_{m-1}}^{E0} \left(\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m} \right) - \frac{1}{6} \left(C_{Bvar_{m-1}}^{E0}\, \Delta\underline{\alpha}_{var}^{Bvar} \right) \times \underline{g}_{avg}^{E0}\, T_m$$

$$\text{For } m = -7, -5, -3, -1: \quad \underline{V}_{var_m}^{E0} = \underline{V}_{ref_0}^{E0} \tag{47}$$

For $m = 1, 3, 5, \cdots$:

$$\underline{V}_{var_m}^{E0} \approx \underline{V}_{ref_0}^{E0} - 2\, C_{Bvar_{m-1}}^{E0} \left[\left(\Delta\underline{\alpha}_{var}^{Bvar} - \Delta\underline{\alpha}'^{Bvar}_{var} \right) \times \frac{\underline{l}^{Bref}}{T_m} \right]$$

$$+ \frac{1}{6} \left[C_{Bvar_{m-1}}^{E0} \left(\Delta\underline{\alpha}_{var}^{Bvar} - \Delta\underline{\alpha}'^{Bvar}_{var} \right) \right] \times \underline{g}_{avg}^{E0}\, T_m$$

For $m = 2, 4, 6, \cdots$:

$$\underline{V}_{var_m}^{E0} \approx \underline{V}_{ref_0}^{E0} + 2\, C_{Bvar_{m-1}}^{E0} \left(\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m} \right) - \frac{1}{6} \left(C_{Bvar_{m-1}}^{E0}\, \Delta\underline{\alpha}_{var}^{Bvar} \right) \times \underline{g}_{avg}^{E0}\, T_m$$

Comparing Method 1 velocity $\underline{V}_{var_m}^{E0}$ in (47) with the equivalent theoretical result in (45), we see that the (46) Method 1 specific force oscillations create equivalent unrealistic cyclic velocity effects, triggered by transitions in angular rate at $m = -9$ and $m = 0$.

For $-9 < m \le 0$, Method 1 velocity $\underline{V}_{var_m}^{E0}$ in (47) oscillates from m cycle to cycle from $\underline{V}_{ref_0}^{E0}$ at even m cycles to $\underline{V}_{ref_0}^{E0} + 2\, C_{Bvar_{m-1}}^{E0} \left(\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m} \right)$ at odd m cycles, averaging (to first order accuracy) the theoretical continuous $\underline{V}_{ref_0}^{E0} + C_{Bvar_{m-1}}^{E0} \left(\Delta\underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m} \right)$ solution in (45). In addition, at only even m cycles, the (47) result contains a $-\frac{1}{6} \left(C_{Bvar_{m-1}}^{E0}\, \Delta\underline{\alpha}_{var}^{Bvar} \right) \times \underline{g}^{E0}\, T_m$ term not present in the theoretical (45) solution. (This term is required to generate a position solution that meets the (37) position constraint).

For $m > 0$, (47) shows a Method 1 velocity profile that oscillates from $\underline{V}_{ref_0}^{E0} - 2\, C_{Bvar_{m-1}}^{E0} \left[\left(\Delta\underline{\alpha}_{var}^{Bvar} - \Delta\underline{\alpha}'^{Bvar}_{var} \right) \times \frac{\underline{l}^{Bref}}{T_m} \right]$ at odd m cycles to

$\underline{V}^{E0}_{ref_0} + 2\, C^{E0}_{Bvar_{m-1}} \left(\Delta\underline{\alpha}^{Bvar}_{var} \times \dfrac{\underline{l}^{Bref}}{T_m} \right)$ at even m cycles, plus gravity terms not present in theoretical (45): $+\dfrac{1}{6}\left[C^{E0}_{Bvar_{m-1}} \left(\Delta\underline{\alpha}^{Bvar}_{var} - \Delta\underline{\alpha}'^{Bvar}_{var} \right) \right] \times \underline{g}^{E0}\, T_m$ at odd m cycles and $-\dfrac{1}{6}\left(C^{E0}_{Bvar_{m-1}}\, \Delta\underline{\alpha}^{Bvar}_{var} \right) \times \underline{g}^{E0}\, T_m$ at even m cycles.

As for the previous specific force discussion, it is informative to analyze (47) velocity when $\Delta\underline{\alpha}'^{Bvar}_{var} = \Delta\underline{\alpha}^{Bvar}_{var}$ and when $\Delta\underline{\alpha}'^{Bvar}_{var} = 0$. For $\Delta\underline{\alpha}'^{Bvar}_{var} = \Delta\underline{\alpha}^{Bvar}_{var}$, the $\left(\Delta\underline{\alpha}^{Bvar}_{var} - \Delta\underline{\alpha}'^{Bvar}_{var} \right)$ terms vanish (for $m > 0$) producing an unrealistic velocity that oscillates from $\underline{V}^{E0}_{ref_0}$ at odd m cycles to $\underline{V}^{E0}_{ref_0} + 2\, C^{E0}_{Bvar_{m-1}} \left(\Delta\underline{\alpha}^{Bvar}_{var} \times \dfrac{\underline{l}^{Bref}}{T_m} \right) - \dfrac{1}{6}\left(C^{E0}_{Bvar_{m-1}}\, \Delta\underline{\alpha}^{Bvar}_{var} \right) \times \underline{g}^{E0}_{avg}\, T_m$ at even m cycles, i.e., an extension of the Method 1 result in (47) for $-9 < m \le 0$. When $\Delta\underline{\alpha}'^{Bvar}_{var} = 0$, the (47) Method 1 solution for $m > 0$ matches the theoretical (45) $\underline{V}^{E0}_{ref_0}$ result, plus a sustained $2\, C^{E0}_{Bvar_{m-1}} \left(\Delta\underline{\alpha}^{Bvar}_{var} \times \dfrac{\underline{l}^{Bref}}{T_m} \right) - \dfrac{1}{6}\left(C^{E0}_{Bvar_{m-1}}\, \Delta\underline{\alpha}^{Bvar}_{var} \right) \times \underline{g}^{E0}_{avg}\, T_m$ oscillation, clearly an unrealistic result.

METHOD 1 POSITION

Method 1 is designed to satisfy position constraint (6) with (7). For the test example, (37) replaces (7), hence, Method 1 position for the test example becomes:

$$\begin{aligned} &\text{For } m \le -9: \quad \underline{R}^{E0}_{var_m} = m\, \underline{V}^{E0}_{ref_0}\, T_m + C^{E0}_{Bvar_{-9}}\, \underline{l}^{Bref} \\ &\text{For } m > -9: \quad \underline{R}^{E0}_{var_m} = m\, \underline{V}^{E0}_{ref_0}\, T_m + C^{E0}_{Bvar_m}\, \underline{l}^{Bref} \end{aligned} \tag{48}$$

For the Method 1 approach, (48) is implicitly incorporated in (46) – (47) at each m cycle, as can be confirmed by substituting (46) – (47) in (4) with (6) and (15. Thus, (48) for $\underline{R}^{E0}_{var_m}$ exactly matches the (45) theoretical result.

METHOD 1 PERFORMANCE SUMMARY

The sustained additional lever arm oscillatory components present in (46) and (47) illustrate the fundamental problem using Method 1 for a variation trajectory generator: creating unrealistic sustained oscillations in specific force and velocity. It is informative to also note for the test example, that if the $\underline{\omega}^{Bvar}$ to $\underline{\omega}'^{Bvar}$ change in angular rate occurred one m cycle earlier (or later), the (46) – (47) solution following that change would be of the same form as following the $m = -9$ change, but with $\Delta\underline{\alpha}_{var}^{Bvar}$ replaced by $\Delta\underline{\alpha}'^{Bvar}_{var}$. Thus, for the Method 1 approach in general, each transition in angular rate for the variation trajectory will trigger another sustained oscillation that may add to or subtract from those generated previously, depending on the m cycle phasing, the vector direction of each integrated angular rate increment component, and the value of $C_{E0_{m-1}}^{Bvar}$ attitude.

The Method 2 solution discussed next eliminates the Method 1 oscillation effect for lever arm terms, but not completely for gravity, an unavoidable consequence of structuring the variation trajectory from a sequence of constant integrated angular-rate/specific-force m cycles.

METHOD 2 PERFORMANCE UNDER THE TEST EXAMPLE

The following subsections present and discuss Method 2 performance results derived in [2, Appendix H] under test example conditions using (31) for specific force increment $\Delta\underline{\upsilon}_{var_m}^{Bvar}$, with substitution in (3) - (4) for $\underline{V}_{var_m}^{E0}$, $\underline{R}_{var_m}^{E0}$ velocity/position response.

METHOD 2 SPECIFIC FORCE

The [2, Appendix H] derived (H-45) result for Method 2 specific force under test example conditions is renumbered and shown next.

For $n < -4$:

$$\Delta \underline{\upsilon}_{var_{2n-1}}^{Bvar} = -C_{E0_{-9}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \qquad \Delta \underline{\upsilon}_{var_{2n}}^{Bvar} = -C_{E0_{-9}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m$$

For $n = -4$:

$$\Delta \underline{\upsilon}_{var_{-9}}^{Bvar} = \frac{1}{2}\Delta \underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m} - \left[I - \frac{1}{4}\left(\Delta \underline{\alpha}_{var}^{Bvar} \times\right)\right] C_{E0_{-9}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m - \frac{1}{3}\Delta \underline{\alpha}_{var}^{Bvar} \times \left(C_{E0_{-9}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$$

$$\Delta \underline{\upsilon}_{var_{-8}}^{Bvar} = \frac{1}{2}\Delta \underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m} - \left[I - \frac{1}{4}\left(\Delta \underline{\alpha}_{var}^{Bvar} \times\right)\right] C_{E0_{-9}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m + \frac{1}{3}\Delta \underline{\alpha}_{var}^{Bvar} \times \left(C_{E0_{-9}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right)$$

For $-4 < n < 0$:

$$\begin{aligned} \Delta \underline{\upsilon}_{var_{2n-1}}^{Bvar} = {} & \Delta \underline{\alpha}_{var}^{Bvar} \times \left(\Delta \underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m} \right) - \left[I - \frac{1}{2}\left(\Delta \underline{\alpha}_{var}^{Bvar} \times\right)\right] C_{E0_{2n-2}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \\ & - \frac{1}{6}\Delta \underline{\alpha}_{var}^{Bvar} \times \left(C_{E0_{2n-2}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right) \\ \Delta \underline{\upsilon}_{var_{2n}}^{Bvar} = {} & \Delta \underline{\alpha}_{var}^{Bvar} \times \left(\Delta \underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m} \right) - \left[I - \frac{1}{2}\left(\Delta \underline{\alpha}_{var}^{Bvar} \times\right)\right] C_{E0_{2n-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \\ & + \frac{1}{6}\Delta \underline{\alpha}_{var}^{Bvar} \times \left(C_{E0_{2n-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right) \end{aligned} \tag{49}$$

For $n = 0$:

$$\begin{aligned} \Delta \underline{\upsilon}_{var_{-1}}^{Bvar} = {} & -\frac{1}{4}\left(\Delta \underline{\alpha}'^{Bvar}_{var} - \Delta \underline{\alpha}_{var}^{Bvar}\right) \times \frac{\underline{l}^{Bref}}{T_m} - \left[I - \frac{1}{2}\left(\Delta \underline{\alpha}_{var}^{Bvar} \times\right)\right] C_{E0_{-2}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \\ & - \frac{1}{6}\Delta \underline{\alpha}_{var}^{Bvar} \times \left(C_{E0_{-2}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right) \\ \Delta \underline{\upsilon}_{var_{0}}^{Bvar} = {} & \frac{3}{4}\left(\Delta \underline{\alpha}'^{Bvar}_{var} - \Delta \underline{\alpha}_{var}^{Bvar}\right) \times \frac{\underline{l}^{Bref}}{T_m} - \left[I - \frac{1}{2}\left(\Delta \underline{\alpha}_{var}^{Bvar} \times\right)\right] C_{E0_{-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \\ & + \frac{1}{6}\Delta \underline{\alpha}_{var}^{Bvar} \times \left(C_{E0_{-1}}^{Bvar}\, \underline{g}_{avg}^{E0}\, T_m \right) \end{aligned}$$

(Continued)

(49) Concluded

For $n = 1$:

$$\Delta \underline{\upsilon}_{var_1}^{Bvar} = \frac{3}{4}\left(\Delta \underline{\alpha}'^{Bvar}_{var} - \Delta \underline{\alpha}^{Bvar}_{var}\right) \times \frac{\underline{l}^{Bref}}{T_m} - \left[I - \frac{1}{2}\left(\Delta \underline{\alpha}'^{Bvar}_{var} \times\right)\right] C^{Bvar}_{E0_0} \, \underline{g}^{E0}_{avg} \, T_m$$
$$- \frac{1}{6}\Delta \underline{\alpha}'^{Bvar}_{var} \times \left(C^{Bvar}_{E0_0} \, \underline{g}^{E0}_{avg} \, T_m \right)$$

$$\Delta \underline{\upsilon}_{var_2}^{Bvar} = -\frac{1}{4}\left(\Delta \underline{\alpha}'^{Bvar}_{var} - \Delta \underline{\alpha}^{Bvar}_{var}\right) \times \frac{\underline{l}^{Bref}}{T_m} - \left[I - \frac{1}{2}\left(\Delta \underline{\alpha}'^{Bvar}_{var} \times\right)\right] C^{Bvar}_{E0_1} \, \underline{g}^{E0}_{avg} \, T_m$$
$$+ \frac{1}{6}\Delta \underline{\alpha}'^{Bvar}_{var} \times \left(C^{Bvar}_{E0_1} \, \underline{g}^{E0}_{avg} \, T_m \right)$$

For $n > 1$:

$$\Delta \underline{\upsilon}_{var_{2n-1}}^{Bvar} = \Delta \underline{\alpha}'^{Bvar}_{var} \times \left(\Delta \underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m} \right) - \left[I - \frac{1}{2}\left(\Delta \underline{\alpha}'^{Bvar}_{var} \times\right)\right] C^{Bvar}_{E0_{2n-2}} \, \underline{g}^{E0}_{avg} \, T_m$$
$$- \frac{1}{6}\Delta \underline{\alpha}'^{Bvar}_{var} \times \left(C^{Bvar}_{E0_{2n-2}} \, \underline{g}^{E0}_{avg} \, T_m \right)$$

$$\Delta \underline{\upsilon}_{var_{2n}}^{Bvar} = \Delta \underline{\alpha}'^{Bvar}_{var} \times \left(\Delta \underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m} \right) - \left[I - \frac{1}{2}\left(\Delta \underline{\alpha}'^{Bvar}_{var} \times\right)\right] C^{Bvar}_{E0_{2n-1}} \, \underline{g}^{E0}_{avg} \, T_m$$
$$+ \frac{1}{6}\Delta \underline{\alpha}'^{Bvar}_{var} \times \left(C^{Bvar}_{E0_{2n-1}} \, \underline{g}^{E0}_{avg} \, T_m \right)$$

<u>Method 2 Specific Force Response To Lever Arm Effects</u>

From (49) (while recognizing that $m = 2n$), we see that for $n < -4$, $-4 < n < 0$, and $n > 1$ the Method 2 generated integrated specific force increment $\Delta \underline{\upsilon}_{var_m}^{Bvar}$ response to the $\underline{l}^{Bref}$ lever arm effect exactly matches the (45) theoretical solution. This contrasts with the Method 1 result in (46) that generates unrealistic oscillatory terms during the $-4 < n < 0$, and $n > 1$ rotation time periods. Coupled with the (49) response, Method 2 creates a lever arm effect that transitions specific force during $\underline{\omega}_{var}^{Bvar}$ rotation rate change: from zero to $\underline{\omega}^{Bvar}$ at $m = -9$ and from $\underline{\omega}^{Bvar}$ to $\underline{\omega}'^{Bvar}$ at $m = 0$. For the $m = -9$ change, the transient spans m cycles -9 and -8, for the $m = 0$ change, the transient spans m cycles -1 to 2. The transient accommodates the finite width/amplitude of the (49) specific force increments compared to the instantaneous impulsive increments required in the theoretical (45) solution for instantaneous angular rate transition. Transients are also present in the (46) Method 1 response during angular rate change, but these are also the source of sustained oscillations.

Method 2 Specific Force Response To Gravity

The theoretical solution in (45) contains a gravity component $C_{E0_{m-1/2}}^{Bvar} \underline{g}_{avg}^{E0} T_m$ during rotation to balance gravity. Eqs. (49) show for Method 2 (as for Method 1) that such a component is also present during rotation periods as represented by the equivalent $\left[I - \frac{1}{2}\left(\Delta\underline{\alpha}_{var}^{Bvar} \times\right)\right]\left(C_{E0_{m-1}}^{Bvar} \underline{g}_{avg}^{E0} T_m\right)$ for $-4 < n < 0$ and $\left[I - \frac{1}{2}\left(\Delta\underline{\alpha}'^{Bvar}_{var} \times\right)\right]\left(C_{E0_{m-1}}^{Bvar} \underline{g}_{avg}^{E0} T_m\right)$ for $n > 1$. As with Method 1, during rotation, Method 2 also contains oscillatory terms; $\frac{1}{6}\Delta\underline{\alpha}_{var}^{Bvar} \times \left(C_{E0_{-1}}^{Bvar} \underline{g}_{avg}^{E0} T_m \right)$ during $-4 < n < 0$ and $\frac{1}{6}\Delta\underline{\alpha}'^{Bvar}_{var} \times \left(C_{E0_1}^{Bvar} \underline{g}_{avg}^{E0} T_m \right)$ during $n > 1$. These are the unavoidable consequences of requiring the specific force increments for the variation trajectory generator to be based on constant acceleration over each *m* cycle (versus the theoretical (45) solution that allows continuous specific force in time). Unlike Method 1, Method 2 specific force in (49) contains a gravity effect transient spanning *m* cycles -9 to -8 when angular rate changes from zero to $\underline{\omega}^{Bvar}$.

METHOD 2 VELOCITY

Having found $\Delta\underline{\upsilon}_{var_{m-1}}^{Bvar}$, $\Delta\underline{\upsilon}_{var_m}^{Bvar}$ in (49), the Velocity Determination section of [2, Appendix H] derives the corresponding velocity history by substitution in (3). The resulting Method 2 velocity under test example conditions is then provided to first order accuracy by (H-53) of [2, Appendix H] as renumbered and shown next.

For $n < -4$:

$$\underline{V}^{E0}_{var\,2n-1} = \underline{V}^{E0}_{ref_0} \qquad \underline{V}^{E0}_{var\,2n} = \underline{V}^{E0}_{ref_0}$$

For $n = -4$:

$$\underline{V}^{E0}_{var-9} = \underline{V}^{E0}_{ref_0} + \frac{1}{2} C^{E0}_{Bvar-9}\left(\Delta\underline{\alpha}^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m}\right) - \frac{1}{12}\left(C^{E0}_{Bvar-9}\,\Delta\underline{\alpha}^{Bvar}_{var}\right) \times \underline{g}^{E0}_{avg}\, T_m$$

$$\underline{V}^{E0}_{var-8} = \underline{V}^{E0}_{ref_0} + C^{E0}_{Bvar-9}\left(\Delta\underline{\alpha}^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m}\right)$$

For $-4 < n < 0$:

$$\underline{V}^{E0}_{var\,2n-1} = \underline{V}^{E0}_{ref_0} + C^{E0}_{Bvar\,2n-2}\left(\Delta\underline{\alpha}^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m}\right) - \frac{1}{6}\left(C^{E0}_{Bvar\,2n-2}\,\Delta\underline{\alpha}^{Bvar}_{var}\right) \times \underline{g}^{E0}_{avg}\, T_m$$

$$\underline{V}^{E0}_{var\,2n} = \underline{V}^{E0}_{ref_0} + C^{E0}_{Bvar\,2n-1}\left(\Delta\underline{\alpha}^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m}\right)$$

For $n = 0$:

$$\underline{V}^{E0}_{var-1} = \underline{V}^{E0}_{ref_0} + C^{E0}_{Bvar-2}\left(\Delta\underline{\alpha}^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m}\right) + \frac{1}{4} C^{E0}_{Bvar-2}\left[\left(\Delta\underline{\alpha}^{Bvar}_{var} - \Delta\underline{\alpha}'^{Bvar}_{var}\right) \times \frac{\underline{l}^{Bref}}{T_m}\right]$$
$$-\frac{1}{6}\left(C^{E0}_{Bvar-2}\,\Delta\underline{\alpha}^{Bvar}_{var}\right) \times \underline{g}^{E0}_{avg}\, T_m \qquad (50)$$

$$\underline{V}^{E0}_{var\,0} = \underline{V}^{E0}_{ref_0} + C^{E0}_{Bvar\,0}\left(\Delta\underline{\alpha}^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m}\right) + \frac{1}{2} C^{E0}_{Bvar-1}\left[\left(\Delta\underline{\alpha}'^{Bvar}_{var} - \Delta\underline{\alpha}^{Bvar}_{var}\right) \times \frac{\underline{l}^{Bref}}{T_m}\right]$$

For $n = 1$:

$$\underline{V}^{E0}_{var_1} = \underline{V}^{E0}_{ref_0} + C^{E0}_{Bvar\,0}\left(\Delta\underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m}\right) + \frac{1}{4} C^{E0}_{Bvar\,0}\left[\left(\Delta\underline{\alpha}'^{Bvar}_{var} - \Delta\underline{\alpha}^{Bvar}_{var}\right) \times \frac{\underline{l}^{Bref}}{T_m}\right]$$
$$-\frac{1}{6}\left(C^{E0}_{Bvar\,0}\,\Delta\underline{\alpha}'^{Bvar}_{var}\right) \times \underline{g}^{E0}_{avg}\, T_m$$

$$\underline{V}^{E0}_{var\,2} = \underline{V}^{E0}_{ref_0} + C^{E0}_{Bvar_1}\left(\Delta\underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m}\right)$$

For $n > 1$:

$$\underline{V}^{E0}_{var\,2n-1} = \underline{V}^{E0}_{ref_0} + C^{E0}_{Bvar\,2n-2}\left(\Delta\underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m}\right) - \frac{1}{6}\left(C^{E0}_{Bvar\,2n-2}\,\Delta\underline{\alpha}'^{Bvar}_{var}\right) \times \underline{g}^{E0}_{avg}\, T_m$$

$$\underline{V}^{E0}_{var\,2n} = \underline{V}^{E0}_{ref_0} + C^{E0}_{Bvar\,2n-1}\left(\Delta\underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m}\right)$$

Comparing the (50) velocity results at "even" m cycles (i.e., $m = 2n$) with (H-21) of [2, Appendix H] (at each n cycle) shows identical results within the first order derivation approximation accuracy. This confirms the first order accuracy of (31) for Method 2 in generating a specific force profile that meets the specified velocity constraints in (10) at each even m cycle, subject to the (37) – (38) test example conditions. Note, however, that the (50) result at even cycle $m = 0$ differs from the theoretical (45), while agreeing with (H-21) velocity at this time point. The reason is that (45) is based on the theoretical exact velocity versus the (H-21) result based on the (10) velocity approximation. Since (H-21) derives from specific force in (31) subject to the (10) velocity approximation, we should have expected (H-21) to be compatible with (10) velocity, not the theoretical (45) value.

Since the Method 2 approach only controls to meet velocity constraint (10) at even m cycles, we should expect a variation from the (H-21) example test result at odd m cycles (i.e., for $m = 2n - 1$). This is manifested for lever arm effects in (50) as a transient during angular rate change periods, and as a gravity term addition throughout the profile. Both are generated by the Method 2 specific force variations in (46) compared to the theoretical (45) specific force solution.

A lever arm transient effect manifests in (50) for $m = -9$ as $\frac{1}{2} C_{Bvar_{-9}}^{E0} \left(\Delta \underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m} \right)$, the tangential velocity change from zero to the steady $\underline{\omega} \times \underline{l}^{Bref} = \Delta \underline{\alpha}_{var}^{Bvar} \times \frac{\underline{l}^{Bref}}{T_m}$ value at $m = -8$ (the theoretical (45) velocity solution). Similarly, a $\left(\Delta \underline{\alpha}'^{Bvar}_{var} - \Delta \underline{\alpha}_{var}^{Bvar} \right) \times \frac{\underline{l}^{Bref}}{T_m}$ lever arm transient appears in (50) from $m = -1$ to $m = 1$ reflecting the theoretical (45) change in tangential velocity at $m = 0$ from $\underline{\omega} \times \underline{l}^{Bref}$ to $\underline{\omega}' \times \underline{l}^{Bref} = \Delta \underline{\alpha}'^{Bvar}_{var} \times \frac{\underline{l}^{Bref}}{T_m}$. At odd m cycles, the gravity term variation in (50) from the theoretical (45) solution is represented by $\frac{1}{12} \left(C_{Bvar_{-9}}^{E0} \Delta \underline{\alpha}_{var}^{Bvar} \right) \times \underline{g}_{avg}^{E0} T_m$ at $m = -9$, and by $\frac{1}{6} \left(C_{Bvar_{2n-2}}^{E0} \Delta \underline{\alpha}_{var}^{Bvar} \right) \times \underline{g}_{avg}^{E0} T_m$ thereafter.

Notably absent in the (50) Method 2 velocity solution is the sustained lever arm effect oscillation following $m = 1$ generated by the (47) Method 1, triggered by angular rate changes at $m = -9$ and $m = 0$, both retained following their entry.

METHOD 2 POSITION

Having found $\Delta \underline{\upsilon}_{var_{m-1}}^{Bvar}$, $\Delta \underline{\upsilon}_{var_m}^{Bvar}$ in (49) and $\underline{V}_{var_{2n-1}}^{E0}$, $\underline{V}_{var_{2n}}^{E0}$ in (50), the Position Determination section of [2, Appendix H] derives the corresponding position history by substitution into (4) with (3). The resulting Method 2 position under test example conditions is provided to first order accuracy by (H-54) of [2, Appendix H], as renumbered and shown next.

$$\text{For } n < -4:$$

$$\underline{R}_{var\,2n-1}^{E0} = (2n-1)\,\underline{V}_{ref_0}^{E0}\,T_m + C_{Bvar_{-9}}^{E0}\,\underline{l}^{Bref} \qquad \underline{R}_{var\,2n}^{E0} = 2n\,\underline{V}_{ref_0}^{E0}\,T_m + C_{Bvar_{-9}}^{E0}\,\underline{l}^{Bref}$$

$$\text{For } n = -4:$$

$$\underline{R}_{var_{-9}}^{E0} = -9\,\underline{V}_{ref_0}^{E0}\,T_m + C_{Bvar_{-9}}^{E0}\left[I + \frac{1}{4}\left(\Delta\underline{\alpha}_{var}^{Bvar}\times\right)\right]\underline{l}^{Bref} - \frac{1}{24}\left(C_{Bvar_{-9}}^{E0}\,\Delta\underline{\alpha}_{var}^{Bvar}\right)\times\underline{g}_{avg}^{E0}\,T_m^2$$

$$\underline{R}_{var_{-8}}^{E0} = -9\,\underline{V}_{ref_0}^{E0}\,T_m + C_{Bvar_{-8}}^{E0}\,\underline{l}^{Bref}$$

$$\text{For } -4 < n < 0:$$

$$\underline{R}_{var\,2n-1}^{E0} = (2n-1)\,\underline{V}_{ref_0}^{E0}\,T_m + C_{Bvar\,2n-1}^{E0}\,\underline{l}^{Bref} \qquad \underline{R}_{var\,2n}^{E0} = 2n\,\underline{V}_{ref_0}^{E0}\,T_m + C_{Bvar\,2n}^{E0}\,\underline{l}^{Bref}$$

$$\text{For } n = 0: \tag{51}$$

$$\underline{R}_{var_{-1}}^{E0} = -\underline{V}_{ref_0}^{E0}\,T_m + C_{Bvar_{-1}}^{E0}\left\{I - \frac{1}{8}\left[\left(\Delta\underline{\alpha}'^{Bvar}_{var} - \Delta\underline{\alpha}_{var}^{Bvar}\right)\times\right]\right\}\underline{l}^{Bref}$$

$$\underline{R}_{var_0}^{E0} = C_{Bvar_0}^{E0}\,\underline{l}^{Bref}$$

$$\text{For } n = 1:$$

$$\underline{R}_{var_1}^{E0} = \underline{V}_{ref_0}^{E0}\,T_m + C_{Bvar_1}^{E0}\left\{I - \frac{1}{8}\left[\left(\Delta\underline{\alpha}'^{Bvar}_{var} - \Delta\underline{\alpha}_{var}^{Bvar}\right)\times\right]\right\}\underline{l}^{Bref}$$

$$\underline{R}_{var_2}^{E0} = 2\,\underline{V}_{ref_0}^{E0}\,T_m + C_{Bvar_2}^{E0}\,\underline{l}^{Bref}$$

$$\text{For } n > 1:$$

$$\underline{R}_{var\,2n-1}^{E0} = (2n-1)\,\underline{V}_{ref_0}^{E0}\,T_m + C_{Bvar\,2n-1}^{E0}\,\underline{l}^{Bref} \qquad \underline{R}_{var\,2n}^{E0} = 2n\,\underline{V}_{ref_0}^{E0}\,T_m + C_{Bvar\,2n}^{E0}\,\underline{l}^{Bref}$$

From (51) we see that for the test example, Method 2 position generated by integrated specific force increments in (49) and resulting velocity profile in (50), creates a variation trajectory position history that satisfies test example position constraint (37) at all even *m* cycles. This confirms the first order accuracy of (31) for Method 2 in generating a specific force profile that meets the specified (10) position constraint at each even *m* cycle, subject to the (37) – (38) example problem conditions. Interestingly (and unexpectedly), note that under conditions of constant angular rate (i.e., intervals $n < -4$, $-4 < n < 0$, and $n > 1$), Method 2 position in (51) also matches constraint (37) at odd *m* cycles (i.e., for $n = 2m - 1$).

Method 2 position in (51) only deviates from the theoretical (45) value (and the (37) constraint) at odd *m* cycles around $m = -9$ and $m = 0$ during angular rate changes from one constant value to another. Thus, the position response penalty using the Method 2 approach is to generate a transient at odd *m* cycles during angular rate change periods, versus Method 1 position in (48) that exactly satisfies (37) at all *m* cycles due to the (6) – (7) fundamental Method 1 position design constraint.

METHOD 2 RESULTS SUMMARY

The principal advantage of the Method 2 approach is elimination of sustained lever arm and gravity effect specific-force/velocity oscillations present with Method 1: oscillations triggered by angular rate changes that continue throughout the remaining trajectory, the magnitude being substantial, at twice the amplitude of the actual tangential velocity during rotation.

Lever arm transient effects are also present with Method 2, but due to velocity constraint (10), they do not propagate into constant angular rate regions. For most variation trajectory simulations, the number of consecutive constant angular-rate/specific-force intervals is typically on the order of 40, varying in duration from 2 to 3,600 seconds each. For a typical *m* cycle time interval of 0.05 seconds (i.e., 20 Hz), the total time for angular-rate/specific-force transitioning is much less than for sustained angular-rate/specific-force, hence, the transition effect transient for Method 2 is of relatively short time duration.

Method 2, still contains *m* cycle gravity effect specific-force/velocity oscillations, the unavoidable consequence of structuring the variation trajectory generator as a sequence of constant angular-rate/specific-force *m* cycles. The magnitude, however, is fairly small, being on the order of $\frac{1}{6}\Delta\underline{\alpha}_{var}^{Bvar} \times \underline{g}^{E0}\, T_m$ or $\frac{1}{6}\Delta\underline{\alpha}'^{Bvar}_{var} \times \underline{g}^{E0}\, T_m$. For 1 rad/sec angular rate and a 0.05 second *m* cycle time interval T_m (typical), the $\Delta\underline{\alpha}_{var}^{Bvar}$ or $\Delta\underline{\alpha}'^{Bvar}_{var}$ magnitude would be 0.05 rad. Then the magnitude of the velocity oscillation would be on the order of (0.05 × 32.2 × 0.05) / 6 = 0.013 fps. For comparison, consider a lever arm magnitude of 50 ft. Under the hypothesized 1 rad/sec angular rate, the resulting tangential velocity would be 1 × 50 = 50 fps, considerably larger than the 0.013 fps velocity oscillation generated by the gravity effect.

The next section shows how the residual Method 2 gravity effect oscillations in the variation trajectory can be mitigated when applied to Kalman aided INS simulation.

MITIGATING METHOD 2 GRAVITY EFFECT OSCILLATIONS

Residual gravity oscillation effects generated with Method 2 can impact variation trajectory applications in aided INS simulation analyses. When simulating an INS aiding process, the reference trajectory would be used to simulate a “master” INS providing reference data to a Kalman aiding filter. There it would be compared with equivalent navigational data from a simulated INS being aided to form the Kalman filter “measurement” input, Kalman filter outputs then used to update the aided INS.

The variation trajectory angular-rate/specific-force outputs would be used to simulate strapdown gyro/accelerometer inputs to the aided INS; processing the simulated gyro/accelerometer outputs provides the aided INS navigational outputs for Kalman filter measurement. Gravity oscillations on the aided INS navigation data impacts the Kalman filter measurement, hence, Kalman aided INS performance. A simple mitigation of Method 2 gravity oscillation impact on Kalman filter performance is to add the same gravity oscillation to the master INS navigation dat. Then when the revised master and aided INS solutions are compared in the Kalman filter measurement, the gravity oscillations will cancel.

Gravity oscillations are easily added to the master INS output by creating a new reference trajectory from the original, the new version obtained using the same Method 2 approach as for the variation trajectory. The difference would be that the new reference version would use (31) for specific-force based on zero for the (6) and (8) attitude/position offset constraint, i.e., setting $C_{Bvar_m}^{Bref_m}$ to identity for all m cycles, and $\underline{S}_m^{E0}$ to zero for alternate m cycles. The new reference would also impose velocity constraint (10) to go with position constraint (6) at alternate m cycles. Method 2 generated specific force would then be calculated with (31) and applied to the (3) – (4) updating routines to obtain the new reference trajectory. The result will contain the same gravity effect oscillations as the variation trajectory.

Note that the same mitigation technique would implicitly be created in a previous section - Generating Variation And Reference Trajectory Outputs. There, generating reference navigation data using the same integration routines as the variation trajectory, (31) was used to create reference trajectory specific-force (and angular-rate) for integration. The result creates the same gravity effect oscillations present in the variation trajectory, thus (as described previously) cancelling both in the simulated Kalman aided INS measurement.

CONCLUSIONS

Variation trajectories generated using the new Method 2 eliminate sustained lever arm and gravity effect specific-force/velocity oscillations present with the original Method 1 approach. Lever arm transient effects are still present with Method 2, however, due to the added velocity constraint. But they only appear during angular rate changes, and do not propagate as sustained oscillations into the constant angular-rate/specific-force regions. For most variation trajectories the total time for angular-rate transitioning is much less than for constant angular-rate/specific-force, hence, Method 2 lever-arm transients are of relatively short time duration.

Method 2, still contains gravity effect oscillations (as did Method 1), the unavoidable consequence of structuring the variation trajectory as a sequence of constant angular-rate/specific-force update cycles. However, the magnitude is small, and can be effectively mitigated in Kalman aided INS simulation applications. The method is to add the same gravity oscillation to the reference trajectory data. Then when data from the reference and variation trajectories are differenced in the simulated Kalman filter measurement, the gravity oscillations cancel, negating their impact on aided INS performance.

APPENDIX A

INTEGRATED GRAVITY INCREMENTS

Eqs. (3) and (4) contain integrated and doubly integrated gravity increments based on a model that linearly varies in time over an m cycle:

$$\underline{g}_{var}^{E0} = \underline{A} + \underline{B}\left(t - t_{m-1}\right) \tag{A-1}$$

where $\underline{A}$ and $\underline{B}$ are constants that can be defined in terms of $\underline{g}_{var_{m-1}}^{E0}$, $\underline{g}_{var_m}^{E0}$, the $\underline{g}_{var}^{E0}$ values at the $m-1$ and m cycle time instants. From (A-1):

$$\underline{g}_{var_{m-1}}^{E0} = \underline{A} \qquad \underline{g}_{var_m}^{E0} = \underline{A} + \underline{B}\left(t_m - t_{m-1}\right) = \underline{A} + \underline{B}\,T_m \tag{A-2}$$

from which

$$\underline{A} = \underline{g}_{var_{m-1}}^{E0} \qquad \underline{B} = \left(\underline{g}_{var_m}^{E0} - \underline{g}_{var_{m-1}}^{E0}\right) / \, T_m \tag{A-3}$$

Gravity terms in (3) and (4) ($\Delta\underline{V}_{varg}^{E0}$, $\Delta\underline{R}_{varg}^{E0}$) are the integral and double integral of (A-1) over an m cycle:

$$\begin{aligned} \Delta\underline{V}_{varg}^{E0} &= \int_{t_{m-1}}^{t} \underline{g}_{var}^{E0}\, dt = \underline{A}\left(t - t_{m-1}\right) + \underline{B}\,\frac{1}{2}\left(t - t_{m-1}\right)^2 \\ \Delta\underline{R}_{varg}^{E0} &= \int_{t_{m-1}}^{t} \Delta\underline{V}_{var_g}^{E0}\, dt = \underline{A}\,\frac{1}{2}\left(t - t_{m-1}\right)^2 + \underline{B}\,\frac{1}{6}\left(t - t_{m-1}\right)^3 \end{aligned} \tag{A-4}$$

Substituting (A-3) in (A-4) shows that

$$\begin{aligned} \Delta\underline{V}_{varg_m}^{E0} &= \underline{A}\,T_m + \underline{B}\,\frac{1}{2}\,T_m^2 = \underline{g}_{var_{m-1}}^{E0} T_m + \left(\underline{g}_{var_m}^{E0} - \underline{g}_{var_{m-1}}^{E0}\right)\frac{1}{2}\,T_m \\ &= \frac{1}{2}\left(\underline{g}_{var_m}^{E0} + \underline{g}_{var_{m-1}}^{E0}\right) T_m \\ \Delta\underline{R}_{varg_m}^{E0} &= \underline{A}\,\frac{1}{2} T_m^2 + \underline{B}\,\frac{1}{6} T_m^3 = \underline{g}_{var_{m-1}}^{E0}\,\frac{1}{2} T_m^2 + \left(\underline{g}_{var_m}^{E0} - \underline{g}_{var_{m-1}}^{E0}\right)\frac{1}{6} T_m^2 \\ &= \frac{1}{6}\left(\underline{g}_{var_m}^{E0} + 2\,\underline{g}_{var_{m-1}}^{E0}\right) T_m^2 \end{aligned} \tag{A-5}$$

which are the forms appearing in (3) and (4).

APPENDIX B

CALCULATING $\underline{g}_{var_m}^{E0}$ DIRECTLY FROM $\underline{R}_{var_m}^{E0}$

This appendix shows how variation trajectory gravity $\underline{g}_{var_m}^{E0}$ can be computed from $\underline{R}_{var_m}^{E0}$ in (6) based on the analytical development in [1, Fig. 5.2-1 & Sect. 5.4]. For the derivation, the following definitions will apply:

$\underline{R}$ = Position vector from earth's center to navigation point.
$\underline{u}_r$ = Unit vector along $\underline{R}$.
$\underline{u}_{epa}$ = Unit vector along earth polar axis.
$\underline{u}_{eq}$ = Unit vector in earth's equatorial plane lying in the plane defined by $\underline{u}_r$ and $\underline{u}_{epa}$.
$\underline{u}_{\phi}$ = Unit vector perpendicular to $\underline{u}_r$ lying in the plane defined by $\underline{u}_r$ and $\underline{u}_{epa}$.
ϕ = Angle from $\underline{u}_{epa}$ to $\underline{u}_r$.
$\underline{g}$ = Earth's mass attraction gravity vector.
g_r = component of $\underline{g}$ along $\underline{u}_r$.
g_{ϕ} = Component of $\underline{g}$ along $\underline{u}_{\phi}$.

From the previous definitions we first write

$$
\begin{gathered}
\underline{u}_r = \underline{R} / \sqrt{\underline{R} \cdot \underline{R}} \qquad \cos\phi = \underline{u}_{epa} \cdot \underline{u}_r \\
\underline{g} = g_r \, \underline{u}_r + g_{\phi} \, \underline{u}_{\phi} \\
\underline{u}_{\phi} = \cos\phi \, \underline{u}_{eq} - \sin\phi \, \underline{u}_{epa} \\
\underline{u}_{eq} = \left[\underline{u}_r - \left(\underline{u}_{epa} \cdot \underline{u}_r \right) \underline{u}_{epa} \right] / \left| \underline{u}_r - \left(\underline{u}_{epa} \cdot \underline{u}_r \right) \underline{u}_{epa} \right|
\end{gathered} \tag{B-1}
$$

The absolute value parameter in (B-1) can be further defined from

$$
\begin{aligned}
&\left| \underline{u}_r - \left(\underline{u}_{epa} \cdot \underline{u}_r \right) \underline{u}_{epa} \right| = \sqrt{ \left[\underline{u}_r - \left(\underline{u}_{epa} \cdot \underline{u}_r \right) \underline{u}_{epa} \right] \cdot \left[\underline{u}_r - \left(\underline{u}_{epa} \cdot \underline{u}_r \right) \underline{u}_{epa} \right] } \\
&= \sqrt{ 1 - 2 \left(\underline{u}_{epa} \cdot \underline{u}_r \right)^2 + \left(\underline{u}_{epa} \cdot \underline{u}_r \right)^2 } = \sqrt{ 1 - \left(\underline{u}_{epa} \cdot \underline{u}_r \right)^2 } = \sqrt{1 - \cos^2\phi} = \sin\phi
\end{aligned} \tag{B-2}
$$

so that $\underline{u}_{eq}$ in (B-1) simplifies to

$$
\underline{u}_{eq} = \left[\underline{u}_r - \left(\underline{u}_{epa} \cdot \underline{u}_r \right) \underline{u}_{epa} \right] / \sin\phi \tag{B-3}
$$

Hence, the $\underline{u}_\phi$ expression in (B-1) is equivalently

$$
\begin{aligned}
\underline{u}_\phi &= \cos\phi\,\underline{u}_{eq} - \sin\phi\,\underline{u}_{epa} = \cos\phi\left(\underline{u}_r - \underline{u}_{epa}\cos\phi\right)/\sin\phi - \sin\phi\,\underline{u}_{epa} \\
&= \frac{\cos\phi}{\sin\phi}\,\underline{u}_r - \left(\frac{\cos^2\phi}{\sin\phi} + \sin\phi\right)\underline{u}_{epa}
\end{aligned} \qquad \text{(B-4)}
$$

With (B-3) and (B-4), the $\underline{g}$ expression in (B-1) becomes

$$
\begin{aligned}
\underline{g} &= g_r\,\underline{u}_r + g_\phi\,\underline{u}_\phi = g_r\,\underline{u}_r + g_\phi\left[\frac{\cos\phi}{\sin\phi}\,\underline{u}_r - \left(\frac{\cos^2\phi}{\sin\phi} + \sin\phi\right)\underline{u}_{epa}\right] \\
&= \left(g_r + g_\phi\frac{\cos\phi}{\sin\phi}\right)\underline{u}_r - g_\phi\left(\frac{\cos^2\phi}{\sin\phi} + \sin\phi\right)\underline{u}_{epa} \\
&= \left[g_r + \left(\frac{g_\phi}{\sin\phi}\right)\underline{u}_{epa}\cdot\underline{u}_r\right]\underline{u}_r - \left(\frac{g_\phi}{\sin\phi}\right)\underline{u}_{epa}
\end{aligned} \qquad \text{(B-5)}
$$

Eq. (B-5) is now in a form that can be used to determine $\underline{g}_{var_m}^{E0}$ for the article equations based on the classical oblate earth model in [1, Eq. (5.4-1)] and the previous definitions translated into the basic nomenclature utilized in this article:

$$
\begin{aligned}
&R_{var_m} = \sqrt{\underline{R}_{var_m}^{E0}\cdot\underline{R}_{var_m}^{E0}} \qquad \underline{u}_{Rvar_m}^{E0} = \underline{R}_{var_m}^{E0}/R_{var_m} \\
&g_{r_m} = f_r\left[R_{var_m}, (\cos\phi)_m\right] \qquad \left(\frac{g_\phi}{\sin\phi}\right)_m = f_\phi\left[R_{var_m}, (\cos\phi)_m\right] \\
&\underline{g}_{var_m}^{E0} = \left[g_{r_m} + \left(\frac{g_\phi}{\sin\phi}\right)_m \underline{u}_{epa}^{E0}\cdot\underline{u}_{Rvar_m}^{E0}\right]\underline{u}_{Rvar_m}^{E0} - \left(\frac{g_\phi}{\sin\phi}\right)_m \underline{u}_{epa}^{E0}
\end{aligned} \qquad \text{(B-6)}
$$

where the functionals $f_r\left[R_{var_m}, (\cos\phi)_m\right]$ and $f_\phi\left[R_{var_m}, (\cos\phi)_m\right]$ represent the g_{r_m} and $\left(g_\phi/\sin\phi\right)_m$ expressions in [1, Eq. (5.4-1)].

APPENDIX C

CALCULATING $\underline{g}_{var_m}^{E0}$ AS A FIRST ORDER EXPANSION AROUND $\underline{g}_{ref_m}^{E0}$

For applications where reference gravity is provided (or computed) at the *m* cycle rate in $E0$ coordinates (i.e., $\underline{g}_{ref_m}^{E0}$), variation trajectory gravity $\underline{g}_{var_m}^{E0}$ can be calculated using a first order expansion in position displacement $\delta\underline{R}_{ref_m}^{E0}$ of variation position $\underline{R}_{var_m}^{E0}$ from reference position $\underline{R}_{ref_m}^{E0}$. The expansion is based on a classical Newtonian earth gravity model, treating the earth as a uniform density spherical mass so that reference gravity at $\underline{R}_{ref_m}^{E0}$ approximates as

$$\underline{g}_{ref_m}^{E0} \approx -\frac{\mu}{\left|\underline{R}_{ref_m}^{E0}\right|^3}\underline{R}_{ref_m}^{E0} \tag{C-1}$$

where μ is the universal gravitational constant multiplied by the mass of the earth. Expanding $\underline{g}_{ref_m}^{E0}$ around $\underline{R}_{ref_m}^{E0}$ then finds for $\underline{g}_{var_m}^{E0}$:

$$\underline{g}_{var_m}^{E0} = \underline{g}_{ref_m}^{E0} + \delta\underline{g}_{ref_m}^{E0} \approx \underline{g}_{ref_m}^{E0} + \delta\left(-\frac{\mu}{\left|\underline{R}_{ref_m}^{E0}\right|^3}\underline{R}_{ref_m}^{E0}\right) \tag{C-2}$$

with to first order accuracy:

$$\delta\underline{g}_{ref_m}^{E0} = \delta\left(-\frac{\mu}{\left|\underline{R}_{ref_m}^{E0}\right|^3}\underline{R}_{ref_m}^{E0}\right) = -\mu\frac{\delta\underline{R}_{ref_m}^{E0}}{\left|\underline{R}_{ref_m}^{E0}\right|^3} + 3\mu\frac{\delta\left|\underline{R}_{ref_m}^{E0}\right|}{\left|\underline{R}_{ref_m}^{E0}\right|^4}\underline{R}_{ref_m}^{E0} \tag{C-3}$$

Now for some analytical manipulation. First note that

$$\underline{R}_{ref_m}^{E0} \cdot \underline{g}_{ref_m}^{E0} \approx -\frac{\mu}{\left|\underline{R}_{ref_m}^{E0}\right|^3}\underline{R}_{ref_m}^{E0} \cdot \underline{R}_{ref_m}^{E0} = -\frac{\mu}{\left|\underline{R}_{ref_m}^{E0}\right|^3}\left|\underline{R}_{ref_m}^{E0}\right|^2 \tag{C-4}$$

Thus,

$$-\frac{\mu}{\left|\underline{R}_{ref_m}^{E0}\right|^3} = \frac{\underline{R}_{ref_m}^{E0} \cdot \underline{g}_{ref_m}^{E0}}{\left|\underline{R}_{ref_m}^{E0}\right|^2} \qquad \frac{\mu}{\left|\underline{R}_{ref_m}^{E0}\right|^4} = -\frac{\underline{R}_{ref_m}^{E0} \cdot \underline{g}_{ref_m}^{E0}}{\left|\underline{R}_{ref_m}^{E0}\right|^3} \tag{C-5}$$

Substituting (C-5) in (C-3) obtains

$$
\begin{aligned}
\delta \underline{g}_{ref_m}^{E0} &= -\frac{\mu}{\left|\underline{R}_{ref_m}^{E0}\right|^3} \delta \underline{R}_{ref_m}^{E0} + 3\frac{\mu}{\left|\underline{R}_{ref_m}^{E0}\right|^4} \delta \left|\underline{R}_{ref_m}^{E0}\right| \underline{R}_{ref_m}^{E0} \\
&= \frac{\underline{R}_{ref_m}^{E0} \cdot \underline{g}_{ref_m}^{E0}}{\left|\underline{R}_{ref_m}^{E0}\right|^2} \delta \underline{R}_{ref_m}^{E0} - 3\frac{\underline{R}_{ref_m}^{E0} \cdot \underline{g}_{ref_m}^{E0}}{\left|\underline{R}_{ref_m}^{E0}\right|^3} \delta \left|\underline{R}_{ref_m}^{E0}\right| \underline{R}_{ref_m}^{E0}
\end{aligned} \tag{C-6}
$$

The $\delta \left|\underline{R}_{ref_m}^{E0}\right|$ term in (C-6) derives from $\left|\underline{R}_{ref_m}^{E0}\right|^2 = \underline{R}_{ref_m}^{E0} \cdot \underline{R}_{ref_m}^{E0}$ from which

$$
\delta \left|\underline{R}_{ref_m}^{E0}\right|^2 = \delta\left(\underline{R}_{ref_m}^{E0} \cdot \delta \underline{R}_{ref_m}^{E0}\right) = 2\underline{R}_{ref_m}^{E0} \cdot \delta \underline{R}_{ref_m}^{E0} \tag{C-7}
$$

But directly from the differential of $\left|\underline{R}_{ref_m}^{E0}\right|^2$:

$$
\delta \left|\underline{R}_{ref_m}^{E0}\right|^2 = 2\left|\underline{R}_{ref_m}^{E0}\right| \delta \left|\underline{R}_{ref_m}^{E0}\right| \tag{C-8}
$$

Equating (C-7) - (C-8) and solving for $\delta \left|\underline{R}_{ref_m}^{E0}\right|$ thereby obtains

$$
\delta \left|\underline{R}_{ref_m}^{E0}\right| = \underline{R}_{ref_m}^{E0} \cdot \delta \underline{R}_{ref_m}^{E0} \,/\, \left|\underline{R}_{ref_m}^{E0}\right| \tag{C-9}
$$

Substituting (C-9) in (C-6) then gives

$$\delta \underline{g}_{ref_m}^{E0} = \frac{\underline{R}_{ref_m}^{E0} \cdot \underline{g}_{ref_m}^{E0}}{\left| \underline{R}_{ref_m}^{E0} \right|^2} \delta \underline{R}_{ref_m}^{E0} - 3 \frac{\underline{R}_{ref_m}^{E0} \cdot \underline{g}_{ref_m}^{E0}}{\left| \underline{R}_{ref_m}^{E0} \right|^3} \delta \left| \underline{R}_{ref_m}^{E0} \right| \underline{R}_{ref_m}^{E0}$$

$$= \frac{\underline{R}_{ref_m}^{E0} \cdot \underline{g}_{ref_m}^{E0}}{\left| \underline{R}_{ref_m}^{E0} \right|^2} \delta \underline{R}_{ref_m}^{E0} - 3 \frac{\underline{R}_{ref_m}^{E0} \cdot \underline{g}_{ref_m}^{E0}}{\left| \underline{R}_{ref_m}^{E0} \right|^3} \frac{\underline{R}_{ref_m}^{E0} \cdot \delta \underline{R}_{ref_m}^{E0}}{\left| \underline{R}_{ref_m}^{E0} \right|} \underline{R}_{ref_m}^{E0} \qquad \text{(C-10)}$$

$$= \frac{\underline{R}_{ref_m}^{E0}}{\left| \underline{R}_{ref_m}^{E0} \right|} \cdot \underline{g}_{ref_m}^{E0} \frac{1}{\left| \underline{R}_{ref_m}^{E0} \right|} \left(\delta \underline{R}_{ref_m}^{E0} - 3 \frac{\underline{R}_{ref_m}^{E0}}{\left| \underline{R}_{ref_m}^{E0} \right|} \cdot \delta \underline{R}_{ref_m}^{E0} \frac{\underline{R}_{ref_m}^{E0}}{\left| \underline{R}_{ref_m}^{E0} \right|} \right)$$

Lastly, we define a unit vector along $\underline{R}_{ref_m}^{E0}$ as $\underline{u}_{Rref_m}^{E0} \equiv \underline{R}_{ref_m}^{E0} / \left| \underline{R}_{ref_m}^{E0} \right|$ with which (C-10) substituted in (C-2) becomes

$$\underline{g}_{var_m}^{E0} = \underline{g}_{ref_m}^{E0} + \delta \underline{g}_{ref_m}^{E0}$$

$$= \underline{g}_{ref_m}^{E0} + \frac{\underline{u}_{Rref_m}^{E0} \cdot \underline{g}_{ref_m}^{E0}}{\left| \underline{R}_{ref_m}^{E0} \right|} \left(\delta \underline{R}_{ref_m}^{E0} - 3\, \underline{u}_{Rref_m}^{E0} \cdot \delta \underline{R}_{ref_m}^{E0}\, \underline{u}_{Rref_m}^{E0} \right) \qquad \text{(C-11)}$$

Based on small $\underline{S}_m^{E0}$ relative to $\underline{R}_{ref_m}^{E0}$ in position constraint (6), the $\delta \underline{R}_{ref_m}^{E0}$ perturbation from $\underline{R}_{ref_m}^{E0}$ is equated to $\underline{S}_m^{E0}$ with which (C-11) then assumes the final form:

$$\underline{g}_{var_m}^{E0} = \underline{g}_{ref_m}^{E0} + \frac{\underline{u}_{Rref_m}^{E0} \cdot \underline{g}_{ref_m}^{E0}}{\left| \underline{R}_{ref_m}^{E0} \right|} \left(\underline{S}_m^{E0} - 3\, \underline{u}_{Rref_m}^{E0} \cdot \underline{S}_m^{E0}\, \underline{u}_{Rref_m}^{E0} \right) \qquad \text{(C-12)}$$

APPENDIX D

SMOOTHING SPECIFIED ANGULAR DISPLACEMENT OFFSETS

Specifying an angular rate profile characterized by its integrated attitude offset $C_{Bvar_m}^{Bref}$ can be accomplished using constant angular rates over specified time intervals, each lasting for several m cycle T_m intervals. For example, for $C_{Bvar_m}^{Bref}$ representing the angular orientation of a scanning platform mounted in an aircraft, the angular rate generating $C_{Bvar_m}^{Bref}$ might be represented by zero over one time segment, one constant over the next time segment, another constant over the next time interval, and so on. This, however, produces unrealistic step changes in angular rate at the constant rate interval time junctions. This appendix shows how the specified angular rate segments can be smoothed into a more realistic angular rate profile that retains the specified integrated $C_{Bvar_m}^{Bref}$ offset characteristics (when dynamics dissipate) and the traditional orthogonality/normality characteristics of a direction cosine matrix.

The development begins by first defining

$$G_{Cvar2ref_m} \equiv I + \frac{\sin \Delta\beta_{var_m}^{Bvar}}{\Delta\beta_{var_m}^{Bvar}} \left(\Delta\underline{\beta}_{var_m}^{Bvar} \times \right) + \frac{1 - \cos \Delta\beta_{var_m}^{Bvar}}{\left(\Delta\beta_{var_m}^{Bvar} \right)^2} \left(\Delta\underline{\beta}_{var_m}^{Bvar} \times \right)^2 \qquad \text{(D-1)}$$

$$C_{Bvar_m}^{Bref} = C_{Bvar_{m-1}}^{Bref} \, G_{Cvar2ref_m}$$

where $\Delta\underline{\beta}_{var_m}^{Bvar}$ is a rotation vector representing the change in $C_{Bvar_m}^{Bref}$ over an *m* cycle (analogous to $\Delta\underline{\alpha}_{var_m}^{Bvar}$ in (2) for the change in $C_{Bvar_m}^{E0}$), and $G_{Cvar2ref_m}$ is the direction cosine matrix equivalent to $\Delta\underline{\beta}_{var_m}^{Bvar}$ (analogous to G_{Cvar_m} in (2) for $C_{Bvar_m}^{E0}$ updating). If $C_{Bvar_m}^{Bref}$ is specified directly, the equivalent $\Delta\underline{\beta}_{var_m}^{Bvar}$ change each *m* cycle can be computed from the inverse of (D-1):

$$G_{Cvar2ref_m} = \left(C_{Bvar_{m-1}}^{Bref} \right)^{-1} C_{Bvar_m}^{Bref} \qquad \text{(D-2)}$$

with $\Delta\underline{\beta}_{var_m}^{Bvar}$ then obtained exactly from $G_{Cvar2ref_m}$ using the direction cosine to rotation vector extraction approach described in [1, Sect. 3.2.2.2]. The $C_{Bvar_m}^{Bref}$ matrix in (D-2) can be specified implicitly by a sequence of constant angular rate segments, or more generally, by a sequence of fixed $\Delta\underline{\beta}_{var_m}^{Bvar}$ values. For either approach, $\Delta\underline{\beta}_{var_m}^{Bvar}$ will be obtained. The smoothing operation for $C_{Bvar_m}^{Bref}$ is performed on the $\Delta\underline{\beta}_{var_m}^{Bvar}$ values as described next.

We define $\Delta\underline{\beta}_{Flt/var_m}^{Bvar}$ as the smoothed (filtered) value of $\Delta\underline{\beta}_{var_m}^{Bvar}$ and specify that it satisfy an exactness constraint that under "steady state" conditions (i.e., during a sequence of fixed $\Delta\underline{\beta}_{var_m}^{Bvar}$ values when filter transients have decayed to zero), $\Delta\underline{\beta}_{Flt/var_m}^{Bvar}$ will equal $\Delta\underline{\beta}_{var_m}^{Bvar}$.

Ref. [1, Eq. (17.2.1-4)] translates the exactness constraint into the following requirements for the coefficients of a general linear digital filter

$$a_0 + a_1 + a_2 + \cdots + b_1 + b_2 + \cdots = 1 \tag{D-3}$$

where a_i, b_i are constant coefficients in the linear filter:

$$\begin{aligned} \Delta\underline{\beta}_{Flt/var_m}^{Bvar} = {} & a_0\,\Delta\underline{\beta}_{var_m}^{Bvar} + a_1\,\Delta\underline{\beta}_{var_{m-1}}^{Bvar} + a_2\,\Delta\underline{\beta}_{var_{m-2}}^{Bvar} + \cdots \\ & + b_1\,\Delta\underline{\beta}_{Flt/var_{m-1}}^{Bvar} + b_2\,\Delta\underline{\beta}_{Flt/var_{m-2}}^{Bvar} + \cdots \end{aligned} \tag{D-4}$$

Ref. [1, Eq. (17.2.1-11)] shows that when (D-3) is satisfied, the sum of the $\Delta\underline{\beta}_{Flt/var_m}^{Bvar}$ s will equal the sum of the $\Delta\underline{\beta}_{var_m}^{Bvar}$ s within a bounded difference $\underline{\gamma}_m$ where

$$\underline{\gamma}_m \equiv \sum_{i=1}^{n} \Delta\underline{\beta}_{Flt/var_m}^{Bvar} - \sum_{i=1}^{n} \Delta\underline{\beta}_{var_m}^{Bvar} = \sum_{i=1}^{m} \left(\Delta\underline{\beta}_{Flt/var_i}^{Bvar} - \Delta\underline{\beta}_{var_i}^{Bvar} \right) \tag{D-5}$$

Under that sequence of constant $\Delta\underline{\beta}_{var_m}^{Bvar}$ s, the $\underline{\gamma}_m$ difference will be proportional to $\Delta\underline{\beta}_{var_m}^{Bvar}$:

$$\underline{\gamma}_m = -\frac{\sum_{i=1}^{r_a} i\,a_i + \sum_{i=1}^{r_b} i\,b_i}{\sum_{i=0}^{r_a} i\,a_i} \Delta\underline{\beta}_{var_m}^{Bvar} \tag{D-6}$$

where r_a, r_b are the highest numerical subscript for the filter past value coefficients. Eq. (D-6) can also be expressed in the equivalent form:

$$\underline{\gamma}_m = -\frac{1}{T_m} \Delta\underline{\beta}_{var_m}^{Bvar}\,\tau_{Flt} \quad \rightarrow \quad \tau_{Flt} \equiv \frac{\sum_{i=1}^{r_a} i\,a_i + \sum_{i=1}^{r_b} i\,b_i}{\sum_{i=0}^{r_a} i\,a_i} T_m \tag{D-7}$$

where τ_{Flt} is the filter dynamic response-time defined as the time required under a constant $\Delta\underline{\beta}_{var_m}^{Bvar}$ sequence for the $Bvar$ frame to rotate through $\underline{\gamma}_m$. We also define τ_{Flt} as being positive for a "lag" condition when $\underline{\gamma}_m$ is negative (i.e., when the sum of the $\Delta\underline{\beta}_{Flt/var_m}^{Bvar}$ s is less than the sum of the $\Delta\underline{\beta}_{var_m}^{Bvar}$ s), and as negative under a dynamic "lead" condition (when the sum of the $\Delta\underline{\beta}_{Flt/var_m}^{Bvar}$ s is more than the sum of the $\Delta\underline{\beta}_{var_m}^{Bvar}$ s).

An example of a simple filter that might be used for the smoothing function is the so-called "walking window" filter which has zero for the b_i coefficients and equal values for the a_i coefficients. Ref. [1, Eq. (17.2.1-26)] shows that τ_{Flt} in (D-7) then simplifies to

$$\tau_{Flt} = \frac{r_a}{2} T_m \tag{D-8}$$

If it is desirable to eliminate the (D-8) lag effect in the variation trajectory simulator, a simple expedient would be to time shift the $\Delta\underline{\beta}_{var_m}^{Bvar}$ data input to (D-4) by the τ_{Flt} value in (D-8).

It remains to show, based on the previous discussion, how a filtered direction cosine matrix $C_{Bvar/filt_m}^{Bref}$ can be generated as a replacement for $C_{Bvar_m}^{Bref}$ in constraint (7). The method recommended in [1, Sect. 17.2.1] is to define $C_{Bvar/filt_m}^{Bref}$ as $C_{Bvar_m}^{Bref}$ rotated through the difference between the integrated filtered and unfiltered $Bvar$ angular rates (i.e., by $\underline{\gamma}_m$, the difference between the cumulative $\Delta\underline{\beta}_{Flt/var_m}^{Bvar}$ s and $\Delta\underline{\beta}_{var_m}^{Bvar}$ s). This is achieved through the exact $G_{C\gamma_m}$ direction cosine equivalent representation of $\underline{\gamma}_m$:

$$G_{C\gamma_m} \equiv I + \frac{\sin\gamma_m}{\gamma_m}\left(\underline{\gamma}_m \times\right) + \frac{1-\cos\gamma_m}{\gamma_m^2}\left(\underline{\gamma}_m \times\right)^2 \tag{D-9}$$

with which

$$C_{Bvar/filt_m}^{Bref} = C_{Bvar_m}^{Bref}\, G_{C\gamma_m} \tag{D-10}$$

The $\underline{\gamma}_m$ rotation vector in (D-9) would be calculated with $\Delta\underline{\beta}_{var_m}^{Bvar}$ and $\Delta\underline{\beta}_{Flt/var_m}^{Bvar}$ from (D-2) using the equivalent recursive difference form of (D-5):

$$\underline{\gamma}_m = \underline{\gamma}_{m-1} + \Delta\underline{\beta}_{Flt/var_m}^{Bvar} - \Delta\underline{\beta}_{var_m}^{Bvar} \tag{D-11}$$

Since $\underline{\gamma}_m$ would be bounded by virtue of the (D-3) exactness constraint, (D-10) guarantees that $C_{Bvar/filt_m}^{Bref}$ will be angularly close to $C_{Bvar_m}^{Bref}$, only deviating by $\frac{1}{T_m} \Delta \underline{\beta}_{var_m}^{Bvar} \tau_{Flt}$ in (D-7) during periods of steady $\Delta \underline{\beta}_{var_m}^{Bvar}$ inputs, equaling $C_{Bvar_m}^{Bref}$ under zero angular rate (with no residual from previous higher order error buildup), and having the natural normalization/orthogonalization characteristics of a proper direction cosine matrix. The result will be a smoothed more realistic version of $C_{Bvar_m}^{Bref}$ for use in constraint (7).

APPENDIX E

REFERENCE DATACONVERSION FOR VARIATION TRAJECTORY INPUT AND COMPUTED VARIATION DATA CONVERSION FOR OUTPUT

The variation trajectory computation routines developed in this article are executed in inertially non-rotating E_0 frame coordinates. Thus, reference trajectory inputs to the routines must be provided in the E_0 frame or, if not directly available, computed using appropriate conversion formulas. Similarly, once the variation trajectory routines are executed in E_0 coordinates, results must typically be converted into a more traditional format for output. This appendix describes input/output conversion operations from and into a commonly used format: where attitude is defined relative to locally level wander azimuth coordinates, velocity is defined relative to the rotating earth in the locally level frame, and position is defined by altitude above local earth's and the angular orientation of the local level frame relative to earth fixed reference coordinates.

CALCULATING $C_{Bref_m}^{E_0}$, $\underline{R}_{ref_m}^{E_0}$ FOR INPUT TO THE VARIATION TRAJECTORY ANGULAR RATE AND SPECIFIC FORCE FORMULAS

The angular-rate/specific-force solution for the variation trajectory provided in (8), (9), and (31) with (6) and (10) require $C_{Bref_m}^{E_0}$, $\underline{R}_{ref_m}^{E_0}$ reference trajectory inputs (relative to non-rotating inertial coordinates E_0) that may not be directly available from the reference trajectory data file. This section shows how the required data can be generated by analytical conversion of reference trajectory data provided for each *m* cycle in the form of *B* frame attitude $C_{Bref_m}^{N_m}$ relative to a locally level coordinate frame *N* – e.g., wander azimuth [1, Sect. 4.5], angular orientation of the *N* frame $C_{N_m}^{E_m}$ relative to coordinate frame *E* fixed relative to the rotating earth, and altitude h_m above the local earth surface.

The $C_{Bref_m}^{E0}$ matrix is easily obtained using the direction cosine product chain rule:

$$C_{Bref_m}^{E0} = C_{E_m}^{E0}\, C_{N_m}^{E_m}\, C_{Bref_m}^{N_m} \qquad \text{(E-1)}$$

where, as before, E_0 is defined as the angular orientation of the *E* frame in non-rotating inertial space at trajectory time zero, and $C_{E_m}^{E0}$ defines how at cycle time *m*, earth rate has rotated the *E* frame since time zero. Recognizing that earth's rotation rate vector $\underline{\omega}_e^E$ is constant in the E frame, during time interval mT_m the earth rotation angle would be $\underline{\omega}_e^E\, mT_m$, hence, as in [1, Eq. (3.2.2.1-8)] we can write for $C_{E_m}^{E0}$ in (E-1):

$$C_{E_m}^{E0} = I + \frac{\sin \omega_e m T_m}{\omega_e m T_m}\left(\underline{\omega}_e^E \times\right) m T_m + \frac{1-\cos \omega_e m T_m}{\left(\omega_e m T_m\right)^2}\left(\underline{\omega}_e^E \times\right)^2 \left(m T_m\right)^2 \qquad \text{(E-2)}$$

The $\underline{R}_{ref_m}^{E0}$ position vector then derives directly from $C_{N_m}^{E_m}$ and h_m using [1, Eqs. (5.1-9), (5.1-10) & (5.2.1-1)]:

$$\underline{u}_{up}^{E_m} = C_{N_m}^{E_m}\, \underline{u}_{up}^{N} \qquad u_{up_{epa}} = \underline{u}_{epa}^{E} \cdot \underline{u}_{up}^{E_m}$$

$$R_S' = R_0 \sqrt{1 + u_{up_{epa}}\left[(1-e)^2 - 1\right]} \qquad \underline{R}_S^{E_m} = R_S' \left\{ \underline{u}_{up}^{E_m} + \left[(1-e)^2 - 1\right] \underline{u}_{epa}^{E} \right\} \qquad \text{(E-3)}$$

$$\underline{R}_{ref_m}^{E_m} = \underline{R}_S^{E_m} + \underline{u}_{up}^{E_m}\, h_m$$

Note: $\underline{g}_{ref_m}^{E0}$ (for the Appendix C calculation of $\underline{g}_{var_m}^{E0}$) might be available as a reference trajectory output, or can be calculated as in Appendix B using $\underline{R}_{ref_m}^{E}$ from (E-3) and setting $\underline{R}_{ref_m}^{E0}$ in Appendix B to $\underline{R}_{ref_m}^{E}$ (valid due to gravity symmetry around earth's polar rotation axis).

GENERATING VARIATION TRAJECTORY OUTPUTS

The primary inputs to the variation trajectory generator are integrated angular-rate/specific-force increments generated (with Method 2) from (8) and (31). As (8) and (31) are processed from cycle-to-cycle, variation trajectory attitude/velocity/position would be generated using an appropriate set of updating algorithms: (2), (3), and (4) if outputs are to be provided in non-rotating inertial coordinates, or by an alternative for a different navigation data format, e.g., attitude relative to locally level wander azimuth coordinate frame *N*, velocity relative to the rotating earth in the *N* frame, angular orientation of the *N* frame $C_{N_m}^{E_m}$ relative to coordinate frame E fixed to the rotating earth, and altitude h_m above the local earth surface. Included in the variation trajectory output would be the comparable set of reference trajectory attitude/velocity/position data and the integrated angular-rate/specific-force increments that created it. It is important to recognize, however, that computation routines used to generate variation trajectory outputs must be the same as those used in creating the reference trajectory. This is the only way to assure that the difference between the reference and variation trajectories will accurately reflect attitude/velocity/position constraints (6), (7), (8), and (10) (for Method 2).

If the reference trajectory used in creating (8) and (31) is of the previously described wander azimuth type (using the previous section conversion routines to the E_0 frame), the variation trajectory data would be generated using the identical routines (typically as a parallel operation for reference trajectory generation when creating the (8) and (13) integrated angular-rate/specific-force increments (e.g., [4, Eqs. (3), (8) – (10), (12) & (18)] with, for reference trajectory constant angular-rate/specific-force over each m cycle, coning, sculling, scrolling set to zero and doubly integrated specific-force $\underline{S}_{\upsilon_m}$ in [4, Eq. (12) set to $\frac{1}{2}\Delta\underline{\upsilon}_{ref_m}^{Bref} T_m$). If variation trajectory outputs are to be relative to non-rotating E_0 inertial coordinates, the (2) - (4) algorithms would be used to generate the variation trajectory and a parallel set of reference trajectory E_0 frame outputs. If the reference trajectory data was generated from an unknown set of algorithms, it must be recreated using the same computation routines as for the variation trajectory. This would also generally include deriving a set of compatible integrated angular-rate/specific-force increments for reference trajectory regeneration. Included in the previous options is an initialization requirement. Each option is discussed next based on using the Method 2 approach.

INITIALIZATION WHEN OUTPUTTING DATA IN THE E_0 INERTIAL FRAME

The (2) – (4) routines would be used for reference and variation trajectory generation. Attitude/velocity/position initialization would be as defined by reference trajectory input data, either directly in the E_0 frame or after conversion as described in the previous section. Eqs. (8) and (31) would be used for variation trajectory integrated angular-rate/specific-force. If not provided directly, integrated angular-rate/specific-force increments for the reference trajectory would be generated as in (8) and (31) by setting the (6), (8) and (10) variation constraints to zero (i.e., $\underline{S}_m^{E_0} = 0$, $C_{Bvar_m}^{Bref_m} = I$, $\underline{V}_{var_m}^{E_0} = \underline{V}_{ref_m}^{E_0}$).

INITIALIZATION WHEN OUTPUTTING DATA RELATIVE TO WANDER AZIMUTH COORDINATES

An appropriate set of updating routines would be used for reference and variation trajectory generation, e.g., [4, Eqs. (3), (8) – (10), (12) & (18)]. Integrated angular-rate/specific-force increments would be generated as described in the previous subsection. If reference data is provided in wander azimuth coordinates, reference trajectory initialization would set attitude/velocity/position to the input. If reference data is provided in the E_0 frame, appropriate conversion for initialization would be required for both the reference and variation trajectories as described next.

Reference Trajectory Initialization From Data Provided in the $E0$ Frame

Reference trajectory input position provided in $\underline{R}_{ref_0}^{E0}$ format can be converted to the equivalent altitude and local level *N* frame angular orientation in two steps, 1) Finding initial altitude h_{ref_0} and latitude l_{ref_0} directly from $\underline{R}_{ref_0}^{E0}$, and 2) Setting initial wander angle (an arbitrary quantity not impacting navigation accuracy) to zero ($\alpha_{ref_0} = 0$), then finding initial longitude L_{ref_0} by either of two methods (each described subsequently). The initial azimuth wander $C_{N_{ref_0}}^{E0}$ matrix would then be initialized with $\alpha_{ref_0} = 0$ and the calculated l_{ref_0}, L_{ref_0}.

In Step 1 Initial altitude and latitude are calculated from the simultaneous solution of [1, Eqs. (4.4.2.3-7)]:

$$R_S' = R_{eq}\sqrt{1 + u_{Up_{epa/0}}^2\left[(1-e)^2 - 1\right]} \qquad R_S = R_S'\sqrt{1 + u_{Up_{epa/0}}^2\left[(1-e)^4 - 1\right]}$$

$$h_{ref_0} = -\frac{R_{eq}^2}{R_S'} + \sqrt{\left(\frac{R_{eq}^2}{R_S'}\right)^2 + R_{ref_0}^2 - R_S^2} \qquad u_{Up_{epa/0}} = R_{ref_{epa/0}} \Big/ \left[(1-e)^2 R_S' + h_{ref_0}\right] \tag{E-4}$$

with

$$R_{ref_{epa/0}} = \underline{R}_{ref_0}^{E0} \cdot \underline{u}_{epa}^{E0} \qquad R_{ref_0} = \sqrt{\underline{R}_{ref_0}^{E0} \cdot \underline{R}_{ref_0}^{E0}} \tag{E-5}$$

where in the $E0$ frame, $\underline{u}_{epa}^{E0}$ is a unit vector along earth's polar axis, $R_{ref_{epa/0}}$ is the projection of $\underline{R}_{ref_0}^{E0}$ on $\underline{u}_{epa}^{E0}$, and $u_{Up_{epa/0}}$ is the projection on earth's $\underline{u}_{epa}^{E0}$ polar axis of a unit vector upward in the initial locally level *N* frame, R_{eq} is earth's equatorial radius (2.0925604 E7 ft, and e is the ellipticity of earth's ellipsoidal cross-section shape (1/298.257223563). Solving (E-4) for h_{ref_0} and $u_{Up_{epa/0}}$ requires an iterative process in the computational sequence shown beginning with $u_{Up_{epa/0}}$ set to zero. The result will converge in less than 5 cycles.

It can be verified from [1, Fig. 5.2-1] that $u_{Up_{epa/0}}$ equals the sine of initial reference latitude l_{ref_0}, hence:

$$\sin l_{ref_0} = u_{Up_{epa/0}} \qquad \cos l_{ref_0} = \sqrt{1 - u_{Up_{epa/0}}^2} \tag{E-6}$$

Eqs. (E-6) are based on l_{ref_0} defined to vary between plus and minus π/2, hence, the positive polarity of the square root in (E-6) for $\cos l_{ref_0}$.

For Step 2, finding initial longitude L_{ref_0} is most easily accomplished if reference trajectory initial position location is restricted to non-polar regions, e.g., for $R_{ref_{epa/0}} / R_{ref_0} < 0.99$. Then longitude can be directly extracted from $\underline{R}_{ref_0}^{E0}$ using [1, Eqs. (4.4.2.3-6) & (4.4.2.3-4]:

$$L_{ref_0} = \tan^{-1}\left(R_{ref_{eq/\perp mrd\,/0}} \,/\, R_{ref_{eq/mrd\,/0}}\right) \qquad \text{(E-7)}$$

where $R_{ref_{eq/mrd\,/0}}$ is the initial projection of $\underline{R}_{ref_0}^{E0}$ on the intersection axis of earth's equatorial plane with the Greenwich prime meridian plane and $R_{ref_{eq/\perp mrd\,/0}}$ is the initial component of $\underline{R}_{ref_0}^{E0}$ along an axis in the equatorial plane perpendicular to the previously defined Greenwich-meridian/equatorial-plane intersection. Note: Eq. (E-7) should be properly solved numerically using a four-quadrant arc tangent routine.

With initial latitude l_{ref_0} from (E-4) - (E-6), initial longitude L_{ref_0} from (E-7), and initial wander-angle α_{ref_0} specified at zero, initial $C_{N_{ref_0}}^{E0}$ wander azimuth angular position can be determined for example, using [1, Eqs. (4.4.2.1-2)] (based on an *N* frame having *Z*-axis up, Y-axis north at zero wander-angle, and an *E* frame having *Y* along earth's polar axis with Z in the Greenwich meridian reference plane). Including initial altitude h_{ref_0} from (E-4) completes the initial wander azimuth format position conversion process for the reference trajectory.

Initialization of reference trajectory velocity relative to the earth in the wander azimuth *N* frame $\underline{v}_{ref_0}^{N_{ref_0}}$ can be determined directly from input reference velocity $\underline{V}_{ref_0}^{E0}$ relative to non-rotating E_0 coordinates. The method is to use [1, Eq. (4.3-5)] for the E_0 frame equivalent, then transforming the result to the *N* frame using the previously computed $C_{N_{ref_0}}^{E0}$:

$$\underline{v}_{ref_0}^{N_{ref_0}} = \left(C_{N_{ref_0}}^{E0}\right)^{-1}\left(\underline{V}_{ref_0}^{E0} - \omega_e\,\underline{u}_{epa}^{E0}\times\underline{R}_{ref_0}^{E0}\right) = C_{E0}^{N_{ref_0}}\left(\underline{V}_{ref_0}^{E0} - \omega_e\,\underline{u}_{epa}^{E0}\times\underline{R}_{ref_0}^{E0}\right) \quad \text{(E-7)}$$

where ω_e is earth inertial angular rate magnitude (7.2721150 E-5 rad/sec) directed along earth's polar axis unit vector $\underline{u}_{epa}^{E0}$ in the E_0 frame, and where it is recognized that the inverse of a direction cosine matrix is its transpose.

Initialization of reference trajectory attitude relative to the *N* frame is achieved using input inertially referenced attitude $C_{Bref_0}^{E0}$, the previously found $C_{Nref_0}^{E0}$ matrix, and applying the direction cosine matrix multiplication chain rule:

$$C_{Bref_0}^{Nref_0} = \left(C_{Nref_0}^{E0} \right)^{-1} C_{Bref_0}^{E0} = C_{E0}^{Nref_0} \, C_{Bref_0}^{E0} \qquad \text{(E-8)}$$

Variation Trajectory Initialization

Initialization of the variation trajectory position/velocity/attitude for wander azimuth formatting can be achieved using the same method applied for reference trajectory initialization. The only difference would be that $\underline{R}_{ref_0}^{E0}$, $\underline{V}_{ref_0}^{E0}$, $C_{Bref_0}^{E0}$ would be replaced by the equivalent variation trajectory parameters $\underline{R}_{var0}^{E0}$, $\underline{V}_{var0}^{E0}$, C_{Bvar0}^{E0} to obtain the equivalent wander azimuth result: h_{var0}, $C_{N_{var0}}^{E0}$, $\underline{v}_{var0}^{N_{var0}}$, $C_{Bvar0}^{N_{var0}}$. The $\underline{R}_{var0}^{E0}$, $\underline{V}_{var0}^{E0}$, C_{Bvar0}^{E0} inputs would be calculated as in (6) – (8) and (10).

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