MITIGATING SECOND ORDER ERROR EFFECTS IN LINEAR KALMAN FILTERS USING ADAPTIVE PROCESS AND MEASUREMENT NOISE

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INTRODUCTION

Traditional Kalman filter configurations are based on a linearized model that translates error states being estimated into the Kalman filter input measurement [1-3]. In some applications however, second order errors neglected in the linearization process can impact the measurement, masking measurement sensitivities to error states being estimated, hence generating erroneous error state estimates. This is particularly true during initial Kalman filter operations when the error state uncertainties are typically largest.

This article describes a method for mitigating second order error effects using adaptive process and measurement noise in the error model. The approach is a generalization of a technique employed several years ago at Strapdown Associates, Inc. (SAI) for a Kalman aided inertial navigation system (INS) experiencing heading estimation Kalman gain collapse during initial ground alignment. After the generalized approach is developed analytically, a detailed example is provided showing how it would be applied to estimate strapdown INS attitude and velocity error sources.

In practice, many second order effects are negligible and safely ignored with little loss in accuracy. Some however, may be significant (as in the previously sited example). The methodology described shows how all second order effects can be accounted for with the decision then left to the analyst as to which should be included in the system design. An important point to consider for this decision is the benefit derived by including all second order effects, to better focus accuracy assessment during the software validation process.

This article can be considered as an extension to Chapter 15 of my book *Strapdown Analytics II* [3], the supporting document referenced for background information. References to particular sections and equations are identified within square brackets, e.g., Reference 1, Section x, equation (y): [3 - Sect. xx, (y)].

DEFINITIONS

<u>State</u> - One of a group of interactive parameters forming a process. For example, if the process is inertial navigation of a vehicle, then position, velocity and angular

orientation of the vehicle would represent some of the states, each of which contain component states. The position states would be related to the velocity states because velocity produces changes in position. Angular orientation of the vehicle might be represented by a nine element direction cosine matrix (DCM) relating the angular orientation between a reference coordinate frame and one aligned with vehicle axes. The DCM would change with time due to vehicle angular rotation rate (additional states), and the direction cosine elements would be interrelated among themselves through the so-called normality and orthogonality constraints.

Foreground - The grouping of states that form the process.

Error State - The error in the computed or measured value of a state. The analytical representation of an error state may or may not have a direct correspondence with any particular state, but may represent the error in a group of states. For example, the error in angular orientation represented by DCM states (nine components) is usually represented by a small angle rotation vector (three components), each related to groups of the DCM elements.

Error State Vector - Column matrix whose elements are the error states for a particular process.

Kalman Filter - Computational process within a system computer designed to estimate the error state vector associated with a particular process. The measurement input to the Kalman filter is a numerical comparison between particular states in a process (for which the error states are to be estimated) and the equivalent parameters provided from another process. For example, for a navigation system whose error states are to be estimated by a Kalman Filter, the measurement might consist of a comparison between the system computed velocity states (containing errors) and velocity states provided by another device (that also may contain errors).

$\widehat{(\ )}$ - Computed, "estimated", or measured value of the ( ) parameter (containing errors compared to the actual or true value of the ( ) parameter).

$\widetilde{(\ )}$ - Uncertainty (i.e., the error) in $\widehat{(\ )}$. In general, $\widetilde{(\ )}$ is the difference between $\widehat{(\ )}$ and the true value of parameter ( ).

Vector - Parameter or group of parameters represented by a column matrix and identified with an underbar _.

Control Vector - Column matrix whose elements represent changes to be applied by the Kalman Filter to the $\widehat{(\ )}$ parameter states. In general, the control vector is designed to reduce the magnitude of estimated errors in the $\widehat{(\ )}$ parameter states. When applied by the Kalman filter to the system states, the filter applies an equal adjustment to the estimated error state vector to indicate that the estimated error in the states has been adjusted (generally reduced) by the control vector. Details on how the control vector would be formed and applied is provided in [3 - Sect. 15.1.2]. This article only discusses second order error effect considerations in control vector application.

ERROR STATE DYNAMIC AND MEASUREMENT EQUATIONS

Traditional error state dynamic and measurement equation formulas [3 - (15.1-1) & (15.1-2)] can be expanded to account for second order error effects neglected in linearized Kalman filter design:

$$\dot{\underline{x}} = \hat{A}\,\underline{x} + \hat{G}_P\,\underline{n}_P + \dot{\underline{x}}_{2nd} + \cdots \qquad \underline{z}_n = \hat{H}_n\,\underline{x}_n + \hat{G}_{M_n}\,\underline{n}_{M_n} + \underline{z}_{2nd_n} + \cdots \qquad (1)$$

where

$\widehat{(\ )}$ = ( ) value calculated in the Kalman filter computer using real-time estimated or measured parameters containing errors.

$\underline{x}$ = Error state vector.

A = Error state dynamic matrix whose elements are functions of the state parameters in the process being analyzed.

G_P = Process noise matrix whose elements are functions of the state parameters in the process being analyzed.

$\underline{n}_P$ = Vector of uncorrelated white process noise components.

$\dot{\underline{x}}_{2nd}$ = Contribution to $\dot{\underline{x}}$ from second order error terms (in $\underline{x}$)

n = Subscript indicating parameter value at nth Kalman filter estimation cycle time.

$\underline{z}$ = Measurement vector.

H = Measurement matrix whose elements are functions of the state parameters in the process being analyzed.

G_M = Measurement noise dynamic coupling matrix whose elements are functions of the state parameters in the process being analyzed.

$\underline{n}_M$ = Vector of uncorrelated measurement noise components.

$\underline{z}_{2nd_n}$ = Contribution to $\underline{z}_n$ from second order error terms (in $\underline{x}$)

In general, the individual i components of $\dot{\underline{x}}_{2nd}$ and $\underline{z}_{2nd_n}$ will satisfy:

$$\dot{x}_{2nd_i} = \underline{x}^T\,\hat{\vartheta}_i\,\underline{x} \quad i = 1, N \qquad z_{2nd_{n/i}} = \underline{x}^T\,\hat{\Gamma}_i\,\underline{x} \quad i = 1, N_M \qquad (2)$$

where

$\hat{\vartheta}_i$ = N by N upper diagonal matrix where N is the number of elements in error state vector $\underline{x}$. The elements of $\hat{\vartheta}_i$ are functions of the state parameters in the process being analyzed.

$\hat{\Gamma}_i$ = N_M by N_M upper diagonal matrix where N_M is the number of elements in measurement vector $\underline{z}$. The elements of $\hat{\Gamma}_i$ are functions of the state parameters in the process being analyzed.

Equations (2) can also be written in the equivalent form

$$\dot{\underline{x}}_{2nd_i} = \left(\hat{\vartheta}_i \, \underline{x} \, \underline{x}^T\right)_{Tr} \qquad i = 1, N \qquad\qquad z_{2nd_{n/i}} = \left(\hat{\Gamma}_i \, \underline{x} \, \underline{x}^T\right)_{Tr} \qquad i = 1, N_M \qquad (3)$$

where

Tr = Trace (sum of the diagonal elements) of the associated square matrix.

Note that coefficient parameters in (1) - (3) are identified as $\widehat{(\ )}$ estimated values. This arises from the derivation process used for (1) - (3) for which the second order $\dot{\underline{x}}_{2nd}$ and $\underline{z}_{2nd_n}$ elements become defined. From a practical standpoint, the derivation process should be designed to specifically produce the (1) - (3) form because $\widehat{(\ )}$ parameters are the only ones available in the system computer for Kalman filter implementation. The example at the end of this article illustrates the derivation methodology.

APPROXIMATE REAL WORLD MODEL

The "real world error model" is defined as the analytical model for the actual errors to be used as the reference base for Kalman filter design. For compatibility with the linear nature of a traditional Kalman filter analytical process, the real world error model will approximate the second order error effects in the $\dot{\underline{x}}$ equation in (1) by its value at the start of a time interval from computer cycle m-1 to computer cycle m. The m cycle is the digital computation updating interval for an integration process from one Kalman filter estimation time instant to the next (the latter identified as the n cycle). This second order error approximation is reasonable because second order errors are of significance when the first order errors are large, and the change in the magnitude of the errors over an m cycle is generally much smaller than any unusually large error values. The approximate real world model would then be:

From m-1 to m:

$$\dot{\underline{x}} = \hat{A} \, \underline{x} + \hat{G}_P \, \underline{n}_P + \dot{\underline{x}}_{2nd_{m-1}}$$

$$\dot{x}_{2nd_{(m-1)/i}} = \underline{x}_{m-1}^T \, \hat{\vartheta}_i \, \underline{x}_{m-1} \qquad i = 1, N \qquad (4)$$

$$\text{or} \quad \dot{x}_{2nd_{(m-1)/i}} = \left(\hat{\vartheta}_i \, \underline{x}_{m-1} \, \underline{x}_{m-1}^T\right)_{Tr}$$

At m = n:

$$\underline{z}_n = \hat{H}_n \, \underline{x}_n + \hat{G}_{M_n} \, \underline{n}_{M_n} + \underline{z}_{2nd_n}$$

$$z_{2nd_{n/i}} = \underline{x}_n^T \, \hat{\Gamma}_{n_i} \, \underline{x}_n \qquad i = 1, N_M \qquad (5)$$

$$\text{or} \quad z_{2nd_{n/i}} = \left(\hat{\Gamma}_{n_i} \, \underline{x}_n \, \underline{x}_n^T\right)_{Tr}$$

KALMAN FILTER ESTIMATION WORLD MODEL

Over each m cycle time interval, the Kalman filter bases its equivalent to (4) on its estimate of all contributions to (4) except for the completely unknown white process noise, i.e.,

From m-1 to m:

$$\dot{\hat{\underline{x}}} = \hat{A}\,\hat{\underline{x}} + \dot{\hat{\underline{x}}}_{2nd_{m-1}}$$

$$\dot{\hat{x}}_{2nd_{(m-1)/i}} = \hat{\underline{x}}_{m-1}^{T}\,\hat{\vartheta}_i\,\hat{\underline{x}}_{m-1} \qquad i = 1, N \qquad (6)$$

$$\text{or}\quad \dot{\hat{x}}_{2nd_{(m-1)/i}} = \left(\hat{\vartheta}_i\,\hat{\underline{x}}_{m-1}\,\hat{\underline{x}}_{m-1}^{T}\right)_{Tr}$$

The value of $\hat{\underline{x}}_m$ to start the next m cycle is obtained as the integral of $\dot{\hat{\underline{x}}}$ from m-1 to m with $\hat{\underline{x}}_{m-1}$ as the starting value, similar to [3 - (15.1.1-13) & (15.1.2-7)]:

$$\hat{\underline{x}}_m = \hat{\Phi}(t_m,t_{m-1})\,\hat{\underline{x}}_{m-1} + \int_{t_{m-1}}^{t_m}\hat{\Phi}(t,\tau)\,\dot{\hat{\underline{x}}}_{2nd_{m-1}}\,d\tau$$

$$= \hat{\Phi}(t_m,t_{m-1})\,\hat{\underline{x}}_{m-1} + \left(\int_{t_{m-1}}^{t_m}\hat{\Phi}(t,\tau)\,d\tau\right)\dot{\hat{\underline{x}}}_{2nd_{m-1}} \qquad (7)$$

$$= \hat{\Phi}(t_m,t_{m-1})\,\hat{\underline{x}}_{m-1} + \hat{\underline{v}}(t_m,t_{m-1})\,\dot{\hat{\underline{x}}}_{2nd_{m-1}}$$

$$\hat{\Phi}(t,t_{m-1}) = I + \int_{t_{m-1}}^{t}\hat{A}\,\hat{\Phi}(t,\tau)\,d\tau \qquad \hat{\underline{v}}(t_m,t_{m-1}) \equiv \int_{t_{m-1}}^{t_m}\hat{\Phi}(t_m,\tau)\,d\tau$$

where

$\hat{\Phi}(t_2,t_1)$ = State transition matrix that propagates vector data from general time t_1 to general time t_2.

The Kalman filter adjusts its estimate (the so-called "innovations process") of the error state vector at each Kalman n cycle using an input measurement $\underline{z}_n$ and the filter's expectation of what the measurement should be $\hat{\underline{z}}_n$ (based on its previously computed error state vector estimate) similar to [3 - (15.1.2-8) & (15.1.2-9)]:

At m = n:

$$\hat{\underline{x}}_n(-) = \hat{\underline{x}}_m$$

$$\hat{\underline{x}}_m = 0 \quad \text{To start the next m cycle}$$

$$\hat{\underline{z}}_n = \hat{H}_n\,\hat{\underline{x}}_n(-) + \hat{\underline{z}}_{2nd_n}(-)$$

$$\hat{z}_{2nd_{n/i}}(-) = \hat{\underline{x}}_n^{T}(-)\,\hat{\Gamma}_{n_i}\,\hat{\underline{x}}_n(-) \qquad i = 1, N_M \qquad (8)$$

$$\text{or}\quad \hat{z}_{2nd_{n/i}}(-) = \left(\hat{\Gamma}_{n_i}\,\hat{\underline{x}}_n(-)\,\hat{\underline{x}}_n^{T}(-)\right)_{Tr}$$

$$\hat{\underline{x}}_n(+) = \hat{\underline{x}}_n(-) + K_n \left(\underline{z}_n - \hat{\underline{z}}_n\right)$$
$$\hat{\underline{x}}_{m-1} = \hat{\underline{x}}_n(+) \quad \text{To start the next m cycle} \tag{9}$$

where

K_n = Kalman gain matrix designed to minimize the uncertainty (in a statistical sense) in the estimated error state vector $\hat{\underline{x}}_n$ at completion of update (9).

(-) = Designation for a parameter estimate after completion of a Kalman n cycle, but before updating $\hat{\underline{x}}_n$ (the so-called "a priori" estimate).

(+) = Designation for a parameter estimate immediately following the $\hat{\underline{x}}_n$ update (the so-called "a posteriori" estimate).

KALMAN GAIN MATRIX CALCULATION

The K_n gain matrix is designed to optimally minimize the statistical variance of $\tilde{\underline{x}}_n(+)$ after update (9) as represented by the diagonal elements of the covariance matrix P:

$$P \equiv \mathcal{E}\left(\tilde{\underline{x}}\, \tilde{\underline{x}}^T\right) \tag{10}$$

where

$\mathcal{E}(\ )$ = Expected value operator (i.e., statistical average value).

The $\underline{x}$ uncertainty in (10) is defined as:

$$\tilde{\underline{x}} = \hat{\underline{x}} - \underline{x} \quad \text{or equivalently} \quad \hat{\underline{x}} = \underline{x} + \tilde{\underline{x}} \tag{11}$$

An equation for $\tilde{\underline{x}}_n(+)$ in the a posteriori covariance $P(+) = \mathcal{E}\left(\tilde{\underline{x}}_n(+)\, \tilde{\underline{x}}_n^T(+)\right)$ is derived by first substituting (11) for $\hat{\underline{x}}_n(+)$ in (9), with (5) and (8) for $\underline{z}_n$ and $\hat{\underline{z}}_n$, to obtain

$$\tilde{\underline{x}}_n(+) = \tilde{\underline{x}}_n(-) - K_n \left[\hat{H}_n\, \tilde{\underline{x}}_n(-) + \left(\hat{\underline{z}}_{2nd_n}(-) - \underline{z}_{2nd_n}\right) - \hat{G}_{M_n}\, \underline{n}_{M_n}\right] \tag{12}$$

An expression for the components of $\hat{\underline{z}}_{2nd_n}(-) - \underline{z}_{2nd_n}$ in (12) is found from (5) and (8) by substituting the (11) definitions and expanding:

$$
\begin{aligned}
\hat{z}_{2nd_{n/i}} - z_{2nd_{n/i}} &= \underline{\hat{x}}_n^T \, \hat{\Gamma}_{n_i} \, \underline{\hat{x}}_n - \underline{x}_n^T \, \hat{\Gamma}_{n_i} \, \underline{x}_n \\
&= \underline{\hat{x}}_n^T(-) \, \hat{\Gamma}_{n_i} \, \underline{\hat{x}}_n - \left(\underline{\hat{x}}_n - \underline{\tilde{x}}_n\right)^T \hat{\Gamma}_{n_i} \left(\underline{\hat{x}}_n - \underline{\tilde{x}}_n\right) \\
&= \underline{\hat{x}}_n^T \, \hat{\Gamma}_{n_i} \, \underline{\tilde{x}}_n + \underline{\tilde{x}}_n^T \, \hat{\Gamma}_{n_i} \, \underline{\hat{x}}_n - \underline{\tilde{x}}_n^T \, \hat{\Gamma}_{n_i} \, \underline{\tilde{x}}_n \\
&= \underline{\hat{x}}_n^T \, \hat{\Gamma}_{n_i} \, \underline{\tilde{x}}_n + \underline{\hat{x}}_n^T \, \hat{\Gamma}_{n_i}^T \, \underline{\tilde{x}}_n - \left(\hat{\Gamma}_{n_i} \, \underline{\tilde{x}}_n \, \underline{\tilde{x}}_n^T\right)_{Tr} \\
&= \underline{\hat{x}}_n^T \left(\hat{\Gamma}_{n_i} + \hat{\Gamma}_{n_i}^T\right) \underline{\tilde{x}}_n - \left(\hat{\Gamma}_{n_i} \, \underline{\tilde{x}}_n \, \underline{\tilde{x}}_n^T\right)_{Tr}
\end{aligned} \tag{13}
$$

where in deriving (13) it was recognized that since $\underline{\tilde{x}}_n^T \, \hat{\Gamma}_{n_i} \, \underline{\hat{x}}_n$ is a scalar, it equals its transpose $\underline{\hat{x}}_n^T \, \hat{\Gamma}_{n_i}^T \, \underline{\tilde{x}}_n$. With (13), $\underline{\tilde{x}}_n(+)$ from (12) becomes

$$
\underline{\tilde{x}}_n(+) = \left(I - K_n \, \hat{H}_n^*\right) \underline{\tilde{x}}_n(-) + K_n \left(\hat{G}_{M_n} \, \underline{n}_{M_n} + \Delta\underline{\tilde{z}}_{2nd_n}(-)\right) \tag{14}
$$

in which the individual components of $\hat{H}_n^*$ and $\Delta\underline{\tilde{z}}_{2nd_n}(-)$ are defined as

$$
\hat{H}_{n_i}^* \equiv \hat{H}_{n_i} + \underline{\hat{x}}_n^T \left(\hat{\Gamma}_{n_i} + \hat{\Gamma}_{n_i}^T\right) \qquad \Delta\tilde{z}_{2nd_{n/i}}(-) \equiv \left(\hat{\Gamma}_{n_i} \, \underline{\tilde{x}}_n(-) \, \underline{\tilde{x}}_n^T(-)\right)_{Tr} \tag{15}
$$

For a traditional linear Kalman filter design, the equivalent to (14) would be the same, but without $\Delta\underline{\tilde{z}}_{2nd_n}(-)$, and without the (14) augmentation terms in the $\hat{H}_n^*$ components [3 - (1.5.1.2.1-9)].

The K_n gain matrix in (9) is designed to minimize $P_n(+) = \mathcal{E}\left(\underline{\tilde{x}}_n(+) \, \underline{\tilde{x}}_n^T(+)\right)$. Deriving the equation for $P_n(+)$ minimization is classically achieved by substituting (14) into $\underline{\tilde{x}}_n(+) \, \underline{\tilde{x}}_n^T(+)$, expanding, and taking the expected value [3 - Sect. 15.1.2.1]. In this case, the presence of $\Delta\underline{\tilde{z}}_{2nd_n}(-)$ in (14) introduces an additional complexity that can be mitigated by approximating $\Delta\underline{\tilde{z}}_{2nd_n}(-)$ by its expected (or mean) value $\mathcal{E}\left(\Delta\underline{\tilde{z}}_{2nd_n}(-)\right)$. From (15), the individual components of $\mathcal{E}\left(\Delta\underline{\tilde{z}}_{2nd_n}(-)\right)$ are

$$
\mathcal{E}\left(\Delta\underline{\tilde{z}}_{2nd_{n/i}}(-)\right) = \mathcal{E}\left(\hat{\Gamma}_{n_i} \, \underline{\tilde{x}}_n(-) \, \underline{\tilde{x}}_n^T(-)\right)_{Tr} = \left[\hat{\Gamma}_{n_i} \, \mathcal{E}\left(\underline{\tilde{x}}_n(-) \, \underline{\tilde{x}}_n^T(-)\right)\right]_{Tr} = \left(\hat{\Gamma}_{n_i} \, P_n(-)\right)_{Tr} \tag{16}
$$

In addition, as will be apparent subsequently, the presence of second order terms in the (4) and (6) error state rate equations results in a non-zero value for $\mathcal{E}\left(\underline{\tilde{x}}_n(-)\right)$ in (14) which cannot be ignored when deriving the $P_n(+)$ equation. With these differences from traditional $P_n(+)$ derivations, the previous procedure then finds for $P_n(+)$ from (14) and (16):

$$
\begin{aligned}
P_n(+) = \mathcal{E}\Big\langle & \Big\{ \left(I - K_n \hat{H}^*_n\right) \tilde{\underline{x}}_n(-) + K_n \left[\hat{G}_{M_n} \underline{n}_{M_n} + \mathcal{E}\left(\Delta \tilde{\underline{z}}_{2nd_n}(-)\right) \right] \Big\} \\
& \Big\{ \tilde{\underline{x}}^T_n(-) \left(I - \hat{H}^{*T}_n K^T_n\right) + \left[\underline{n}^T_{M_n} \hat{G}^T_{M_n} + \mathcal{E}\left(\Delta \tilde{\underline{z}}^T_{2nd_n}(-)\right) \right] K^T_n \Big\} \Big\rangle \\
= & \left(I - K_n \hat{H}^*_n\right) \mathcal{E}\left(\tilde{\underline{x}}_n(-)\, \tilde{\underline{x}}^T_n(-)\right) \left[I - \left(\hat{H}^*_n\right)^T K^T_n \right] \\
& + \left(I - K_n \hat{H}^*_n\right) \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta \tilde{\underline{z}}^T_{2nd_n}(-)\right) K^T_n \\
& + K_n \hat{G}_{M_n} \mathcal{E}\left(\underline{n}_{M_n} \underline{n}^T_{M_n}\right) \hat{G}^T_{M_n} K^T_n \\
& + K_n \mathcal{E}\left(\Delta \tilde{\underline{z}}_{2nd_n}(-)\right) \mathcal{E}\left(\tilde{\underline{x}}^T_n(-)\right) \left(I - \hat{H}^{*T}_n K^T_n\right) \\
& + K_n \mathcal{E}\left(\Delta \tilde{\underline{z}}_{2nd_n}(-)\right) \mathcal{E}\left(\Delta \tilde{\underline{z}}^T_{2nd_n}(-)\right) K^T_n \\
= & P_n(-) + K_n \begin{bmatrix} \hat{H}^*_n P_n(-) \hat{H}^{*T}_n + R^*_{M_n} \\ - \hat{H}^*_n\ \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta \tilde{\underline{z}}^T_{2nd_n}(-)\right) \\ - \left[\hat{H}^*_n\ \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta \tilde{\underline{z}}^T_{2nd_n}(-)\right) \right]^T \end{bmatrix} K^T_n \\
& - K_n \left[P_n(-) \hat{H}^{*T}_n - \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta \tilde{\underline{z}}^T_{2nd_n}(-)\right) \right]^T \\
& - \left[P_n(-) \hat{H}^{*T}_n - \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta \tilde{\underline{z}}^T_{2nd_n}(-)\right) \right] K^T_n
\end{aligned} \tag{17}
$$

in which the augmented measurement noise matrix $R^*_{M_n}$ is defined as a function of the traditional measurement noise matrix R_{M_n} by

$$
R_{M_n} \equiv \hat{G}_{M_n} \mathcal{E}\left(\underline{n}_{M_n} \underline{n}^T_{M_n}\right) \hat{G}^T_{M_n} \qquad R^*_{M_n} \equiv R_{M_n} + \mathcal{E}\left(\Delta \tilde{\underline{z}}_{2nd_n}(-)\right) \mathcal{E}\left(\Delta \tilde{\underline{z}}^T_{2nd_n}(-)\right) \tag{18}
$$

Equation (17) is now in classic form for finding the optimal K_n that minimizes $P_n(+)$. Applying the standard minimization process in [3 - Sect. 15.1.2.1] obtains the result:

$$
K_n = \begin{bmatrix} P_n(-) \hat{H}^{*T}_n \\ - \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta \tilde{\underline{z}}^T_{2nd_n}(-)\right) \end{bmatrix} \begin{bmatrix} \hat{H}^*_n P_n(-) \hat{H}^{*T}_n + R^*_{M_n} \\ - \hat{H}^*_n\ \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta \tilde{\underline{z}}^T_{2nd_n}(-)\right) \\ - \left[\hat{H}^*_n\ \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta \tilde{\underline{z}}^T_{2nd_n}(-)\right) \right]^T \end{bmatrix}^{-1} \tag{19}
$$

For a traditional linear Kalman filter design, the equivalent to (19) would be the same, but without the $\mathcal{E}\left(\tilde{\underline{x}}_n(-)\right)$ and $\mathcal{E}\left(\Delta \tilde{\underline{z}}_{2nd_n}(-)\right)$ terms, and without the $\hat{H}^*_n$ and R^*_n higher order augmentation terms in (15) and (18). From (19), K_n determination requires $P_n(-)$ (as in the traditional filter) in addition to $\mathcal{E}\left(\tilde{\underline{x}}_n(-)\right)$ and $\hat{\underline{x}}_n(-)$ for $\hat{H}^*_n$, both requiring an integration process (propagation) over the previous Kalman cycle for evaluation. The propagation equation for obtaining $\hat{\underline{x}}_n(-)$ has been described previously in (7) and (8). The next sections derives equations for calculating $\mathcal{E}\left(\tilde{\underline{x}}_n(-)\right)$ and $P_n(-)$.

The refined updating process also requires the calculation of $\mathcal{E}\left(\tilde{\underline{x}}_n(+)\right)$ resulting from update (9). The equation for this operation is easily found by taking the expected value of (14) which finds

$$\mathcal{E}\left(\tilde{\underline{x}}_n(+)\right) = \left(I - K_n \hat{H}_n^*\right) \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) + K_n \mathcal{E}\left(\Delta\tilde{\underline{z}}_{2nd_n}(-)\right) \tag{20}$$

with $\mathcal{E}\left(\Delta\tilde{\underline{z}}_{2nd_n}(-)\right)$ as given by (16).

<u>Calculating $\mathcal{E}\left(\tilde{\underline{x}}_n(-)\right)$</u>

Evaluating $\mathcal{E}\left(\tilde{\underline{x}}_n(-)\right)$ for (19) is an integration process whereby $\frac{d}{dt}\mathcal{E}\left(\tilde{\underline{x}}\right)$ is integrated over individual m-1 to m intervals with the solution sequentially propagated over successive m cycles between Kalman n cycles (similar to $\hat{\underline{x}}$ in (7)). The equation for $\frac{d}{dt}\mathcal{E}\left(\tilde{\underline{x}}\right)$ is derived based on (11) as the estimated difference between $\dot{\hat{\underline{x}}}$ from (6) and $\dot{\underline{x}}$ from (4):

$$\frac{d}{dt}\mathcal{E}\left(\tilde{\underline{x}}\right) = \hat{A}\,\mathcal{E}\left(\tilde{\underline{x}}\right) + \mathcal{E}\left(\dot{\tilde{\underline{x}}}_{2nd_{m-1}}\right) \tag{21}$$

with

$$\dot{\tilde{\underline{x}}}_{2nd_{m-1}} \equiv \dot{\hat{\underline{x}}}_{2nd_{m-1}} - \dot{\underline{x}}_{2nd_{m-1}} \tag{22}$$

An expression for the individual i components of $\mathcal{E}\left(\dot{\tilde{\underline{x}}}_{2nd_{m-1}}\right)$ in (21) is derived by first substituting $\underline{x}_{m-1}$ based on (11) into the (4) formula for $\dot{\underline{x}}_{2nd_{(m-1)/i}}$ and the (6) formula for $\dot{\hat{\underline{x}}}_{2nd_{(m-1)/i}}$, and expanding (22):

$$\begin{aligned}
\dot{\tilde{\underline{x}}}_{2nd_{(m-1)/i}} &= \dot{\hat{\underline{x}}}_{2nd_{(m-1)/i}} - \dot{\underline{x}}_{2nd_{(m-1)/i}} = \hat{\underline{x}}_{m-1}^T\, \hat{\vartheta}_{m-1_i}\, \hat{\underline{x}}_{m-1} - \underline{x}_n^T\, \vartheta_{m-1_i}\, \underline{x}_{m-1} \\
&= \hat{\underline{x}}_{m-1}^T(-)\, \hat{\vartheta}_{m-1_i}\, \hat{\underline{x}}_{m-1} - \left(\hat{\underline{x}}_{m-1} - \tilde{\underline{x}}_{m-1}\right)^T \hat{\vartheta}_{m-1_i} \left(\hat{\underline{x}}_{m-1} - \tilde{\underline{x}}_{m-1}\right) \\
&= \hat{\underline{x}}_{m-1}^T\, \hat{\vartheta}_{m-1_i}\, \tilde{\underline{x}}_{m-1} + \tilde{\underline{x}}_{m-1}^T\, \hat{\vartheta}_{m-1_i}\, \hat{\underline{x}}_{m-1} - \tilde{\underline{x}}_{m-1}^T\, \hat{\vartheta}_{m-1_i}\, \tilde{\underline{x}}_{m-1} \\
&= \hat{\underline{x}}_{m-1}^T\, \hat{\vartheta}_{m-1_i}\, \tilde{\underline{x}}_{m-1} + \hat{\underline{x}}_{m-1}^T\, \hat{\vartheta}_{m-1_i}^T\, \tilde{\underline{x}}_{m-1} - \left(\hat{\vartheta}_{m-1_i}\, \tilde{\underline{x}}_{m-1}\, \tilde{\underline{x}}_{m-1}^T\right)_{Tr} \\
&= \hat{\underline{x}}_{m-1}^T \left(\hat{\vartheta}_{m-1_i} + \hat{\vartheta}_{m-1_i}^T\right) \tilde{\underline{x}}_{m-1} - \left(\hat{\vartheta}_{m-1_i}\, \tilde{\underline{x}}_{m-1}\, \tilde{\underline{x}}_{m-1}^T\right)_{Tr}
\end{aligned} \tag{23}$$

where in deriving (23) it was recognized that since $\underline{\tilde{x}}^T_{m-1}\,\hat{\vartheta}_{m-1_i}\,\underline{\hat{x}}_{m-1}$ is a scalar, it equals its transpose $\underline{\hat{x}}^T_{m-1}\,\hat{\vartheta}^T_{m-1_i}\,\underline{\tilde{x}}_{m-1}$. Taking the expected value of (23) then finds for the individual i components of $\mathcal{E}\left(\dot{\underline{\tilde{x}}}_{2nd_{m-1}}\right)$ in (21):

$$\mathcal{E}\left(\dot{\underline{\tilde{x}}}_{2nd_{(m-1)/i}}\right) = \underline{\hat{x}}^T_{m-1}\left(\hat{\vartheta}_{m-1_i} + \hat{\vartheta}^T_{m-1_i}\right)\mathcal{E}\left(\underline{\tilde{x}}_{m-1}\right) - \left(\hat{\vartheta}_{m-1_i}\,P_{m-1}\right)_{Tr} \qquad (24)$$

The integral of (21) with (24) over the m-1 to m interval provides the propagated value for $\mathcal{E}\left(\underline{\tilde{x}}_m\right)$ as a function of its starting value at m-1 (similar to (7)):

$$\mathcal{E}\left(\underline{\tilde{x}}(t_m)\right) = \hat{\Phi}(t_m,t_{m-1})\,\mathcal{E}\left(\underline{\tilde{x}}_{m-1}\right) + \underline{\hat{v}}(t_m,t_{m-1})\,\mathcal{E}\left(\dot{\underline{\tilde{x}}}_{2nd_{m-1}}\right) \qquad (25)$$

where $\underline{\hat{v}}(t_m,t_{m-1})$ is defined in (7). When m corresponds to a Kalman update n cycle, $\mathcal{E}\left(\underline{\tilde{x}}_n(-)\right)$ for (19) is set equal to $\mathcal{E}\left(\underline{\tilde{x}}_m\right)$:

$$\mathcal{E}\left(\underline{\tilde{x}}_n(-)\right) = \mathcal{E}\left(\underline{\tilde{x}}_m\right) \quad \text{At n before the Kalman update} \qquad (26)$$

Immediately following the Kalman update, $\mathcal{E}\left(\underline{\tilde{x}}_m\right)$ is reset to $\mathcal{E}\left(\underline{\tilde{x}}_n(+)\right)$ (see (20)):

$$\mathcal{E}\left(\underline{\tilde{x}}_{m-1}\right) = \mathcal{E}\left(\underline{\tilde{x}}_n(+)\right) \quad \begin{array}{c}\text{At n following the Kalman update}\\ \text{to start the next m cycle}\end{array} \qquad (27)$$

<u>Calculating</u> $P_n(-)$

Evaluating $P_n(-)$ for (19) is an integration process whereby $\dot{P}$, the derivative of P, is integrated over individual m-1 to m intervals, with the solution sequentially propagated over successive m cycles between Kalman n cycles (similar to $\underline{\hat{x}}$ in (7)). The $\dot{P}$ derivation begins with the integrated solution for $\underline{\tilde{x}}$ at an infinitesimal time interval dt after time t, where t is defined as a general time instant following update cycle m-1.

The $\underline{\tilde{x}}(t+dt)$ solution is $\underline{\tilde{x}}(t)$ plus the integral of $\dot{\underline{\tilde{x}}}$ over dt, the latter derived by subtracting $\dot{\underline{x}}$ in (4) from $\dot{\underline{\hat{x}}}$ in (6), substituting (23), applying (11), and approximating $\dot{\underline{\tilde{x}}}_{2nd_{m-1}}$ by its expected value (having equation (24) components):

$$\dot{\underline{\tilde{x}}} = \hat{A}\,\underline{\tilde{x}} - \hat{G}_P\,\underline{n}_P + \mathcal{E}\left(\dot{\underline{\tilde{x}}}_{2nd_{m-1}}\right) \qquad (28)$$

With (28), $\underline{\tilde{x}}$ at t + dt is:

$$\underline{\tilde{x}}(t+dt) = \underline{\tilde{x}}(t) + \dot{\underline{\tilde{x}}}\, dt$$
$$= \underline{\tilde{x}}(t) + \hat{A}\, \underline{\tilde{x}}(t)\, dt - \underline{w}_P(t+dt,t) + \mathcal{E}\left(\dot{\underline{\tilde{x}}}_{2nd_{m-1}}\right) dt \approx \underline{\tilde{x}}(t) - \underline{w}_P(t+dt,t) \tag{29}$$
$$\underline{w}_P(t+dt,t) \equiv \int_t^{t+dt} \dot{\underline{w}}_P\, dt \qquad \dot{\underline{w}}_P = \hat{G}_P\, \underline{n}_P$$

From (10), $\dot{P}$ at time t + dt is:

$$\dot{P}(t+dt) = \frac{d}{dt}\,\mathcal{E}\left(\underline{\tilde{x}}(t+dt)\, \underline{\tilde{x}}^T(t+dt)\right) = \mathcal{E}\left(\dot{\underline{\tilde{x}}}\, \underline{\tilde{x}}(t+dt)^T + \underline{\tilde{x}}(t+dt)\, \dot{\underline{\tilde{x}}}^T\right)$$
$$= \mathcal{E}\left(\dot{\underline{\tilde{x}}}\, \underline{\tilde{x}}(t+dt)^T\right) + \mathcal{E}\left(\underline{\tilde{x}}(t+dt)\, \dot{\underline{\tilde{x}}}^T\right) \tag{30}$$

which with $\dot{\underline{\tilde{x}}}$ from (28) and $\underline{\tilde{x}}(t+dt)$ from (29) becomes

$$\dot{P}(t+dt) = \hat{A}\, P(t) + P(t)\, \hat{A}^T + \mathcal{E}\left(\dot{\underline{\tilde{x}}}_{2nd_{m-1}}\right) \mathcal{E}\left(\underline{\tilde{x}}\right)^T + \mathcal{E}\left(\underline{\tilde{x}}\right) \mathcal{E}\left(\dot{\underline{\tilde{x}}}^T_{2nd_{m-1}}\right)$$
$$+ \mathcal{E}\left(\dot{\underline{w}}_P\, \underline{w}_P^T(t+dt,t)\right) + \mathcal{E}\left(\underline{w}_P(t+dt,t)\, \dot{\underline{w}}_P^T\right)$$
$$= \hat{A}\, P(t) + P(t)\, \hat{A}^T + \mathcal{E}\left(\dot{\underline{\tilde{x}}}_{2nd_{m-1}}\right) \mathcal{E}\left(\underline{\tilde{x}}\right)^T + \mathcal{E}\left(\underline{\tilde{x}}\right) \mathcal{E}\left(\dot{\underline{\tilde{x}}}^T_{2nd_{m-1}}\right) \tag{31}$$
$$+ \frac{d}{dt}\,\mathcal{E}\left(\underline{w}_P(t+dt,t)\, \underline{w}_P^T(t+dt,t)\right)$$

The $\frac{d}{dt}\,\mathcal{E}\left(\underline{w}_P(t+dt,t)\, \underline{w}_P^T(t+dt,t)\right)$ process noise term in (31) is further evaluated as follows based on a similar development in [3- Sect. 15.1.2.1.1]:

$$\underline{w}_P(t+dt,t) = \int_t^{t+dt} \hat{G}_P\, \underline{n}_P\, d\tau$$
$$\frac{d}{dt}\,\mathcal{E}\left(\underline{w}_P(t+dt,t)\, \underline{w}_P^T(t+dt,t)\right)$$
$$= \frac{d}{dt}\,\mathcal{E}\left[\left(\int_t^{t+dt} \hat{G}_P(\tau_\alpha)\, \underline{n}_P(\tau_\alpha)\, d\tau_\alpha\right)\left(\int_t^{t+dt} \underline{n}_P^T(\tau_\beta)\, \hat{G}_P^T(\tau_\beta)\, d\tau_\beta\right)\right] \tag{32}$$
$$= \frac{d}{dt}\left[\int_t^{t+dt}\left[\int_t^{t+dt} \hat{G}_P(\tau_\alpha)\, \mathcal{E}\left(\underline{n}_P(\tau_\alpha)\, \underline{n}_P^T(\tau_\beta)\right) \hat{G}_P^T(\tau_\beta)\, d\tau_\alpha\right] d\tau_\beta\right]$$
$$= \int_t^{t+dt} \hat{G}_P(\tau_\alpha)\, \mathcal{E}\left(\underline{n}_P(\tau_\alpha)\, \underline{n}_P^T(\tau_\beta)\right) \hat{G}_P^T(\tau_\beta)\, d\tau_\alpha$$

where τ_α and τ_β are Dummy time parameters. The definition of the white process noise vector $\underline{n}_P$ was for the components to be uncorrelated. Hence, the $\mathcal{E}\left(\underline{n}_P(\tau_\alpha)\, \underline{n}_P^T(\tau_\beta)\right)$ matrix will be diagonal. Furthermore, the properties of white noise are such that the expected value of each $\mathcal{E}\left(\underline{n}_P(\tau_\alpha)\, \underline{n}_P^T(\tau_\beta)\right)$ diagonal element will be zero for $\tau_\alpha \neq \tau_\beta$, with its integrated expected value from t to t + dt in (32) equal to the white noise density for that $\underline{n}_P$ component. Thus, (32) reduces to

$$\frac{d}{dt}\,\mathcal{E}\left(\underline{w}_P(t+dt,t)\;\underline{w}_P^T(t+dt,t)\right) = \hat{G}_P(t)\;Q_{P_{Dens}}(t)\;\hat{G}_P^T\,(t) \tag{33}$$

where

$Q_{P_{Dens}}$ = Diagonal matrix in which each element equals the white noise density for the corresponding element in $\underline{n}_P$.

Substituting (33) in (31) then yields the desired expression for $\dot{P}(t)$

$$\begin{gathered}\dot{P}(t) = \hat{A}\;P(t) + P(t)\;\hat{A}^T + Q^*_{Dens}(t) \\ Q^*_{Dens}(t) \equiv \hat{G}_P(t)\;Q_{P_{Dens}}(t)\;\hat{G}_P^T\,(t) + \mathcal{E}\left(\dot{\tilde{\underline{x}}}_{2nd_{m-1}}\right)\mathcal{E}\left(\tilde{\underline{x}}(t)\right)^T + \mathcal{E}\left(\tilde{\underline{x}}(t)\right)\mathcal{E}\left(\dot{\tilde{\underline{x}}}^T_{2nd_{m-1}}\right)\end{gathered} \tag{34}$$

The $\mathcal{E}\left(\tilde{\underline{x}}(t)\right)$ term in (34) at time t is from (25) with (7):

$$\mathcal{E}\left(\tilde{\underline{x}}(t)\right) = \hat{\Phi}\left(t,t_{m-1}\right)\mathcal{E}\left(\tilde{\underline{x}}_{m-1}\right) + \hat{\underline{v}}(t,t_{m-1})\;\mathcal{E}\left(\dot{\tilde{\underline{x}}}_{2nd_{m-1}}\right) \tag{35}$$

An approximate integration algorithm from m-1 to m for $\dot{P}(t)$ in (34) can now be formulated using the approach in [3 - Sect. 15.1.2.1.1.3]:

$$\begin{gathered}P_m = \Phi\left(t_m,t_m\right)\left(P_m + \frac{1}{2}\,Q^*_m\right)\Phi^T\left(t_m,t_m\right) + \frac{1}{2}\,Q^*_m \\ Q^*_m = \int_{t_{m-1}}^{t_m} Q^*_{Dens}(t)\;dt\end{gathered} \tag{36}$$

When m corresponds to a Kalman n cycle update, $P_n(-)$ for the update is set to the corresponding (36) result:

$$\text{At n before the Kalman update:}\quad P_n(-) = P_m \tag{37}$$

Following the Kalman update at cycle n, P_m is then set to the (17) updated value of P:

$$P_{m-1} = P_n(+) \qquad \begin{array}{c}\text{At n following the Kalman update}\\ \text{to start the next m cycle}\end{array} \tag{38}$$

SUMMARY

Following is a summary of the equations for the revised Kalman filter in the order of computation in the user computer. The equations are a repeat of equations (6) - (9), (15) - (20), (24), (27), and (34) - (38).

Basic Inputs From The "Foreground" To The Kalman Filter: $\hat{A}, \hat{G}_P, \hat{\vartheta}, \hat{\Gamma}, \hat{H}$

m Cycle Initialization Immediately Following The Last n Cycle Update

$$\hat{\underline{x}}_{m-1} = \hat{\underline{x}}_n(+) \tag{39}$$

$$P_{m-1} = P_n(+) \tag{40}$$

$$\mathcal{E}\left(\tilde{\underline{x}}_{m-1}\right) = \mathcal{E}\left(\tilde{\underline{x}}_n(+)\right) \tag{41}$$

Sequential Processing Of m Cycles From n-1 to n:

m Cycle Initialization:

$$\hat{\dot{\underline{x}}}_{2nd(m-1)/i} = \hat{\underline{x}}^T_{m-1}\, \hat{\vartheta}_i\, \hat{\underline{x}}_{m-1} \quad \text{or} \quad \hat{\dot{\underline{x}}}_{2nd(m-1)/i} = \left(\hat{\vartheta}_i\, \hat{\underline{x}}_{m-1}\, \hat{\underline{x}}^T_{m-1}\right)_{Tr} \tag{42}$$

$$\mathcal{E}\left(\tilde{\dot{\underline{x}}}_{2nd(m-1)/i}\right) = \hat{\underline{x}}^T_{m-1} \left(\hat{\vartheta}_{m-1_i} + \hat{\vartheta}^T_{m-1_i}\right) \mathcal{E}\left(\tilde{\underline{x}}_{m-1}\right) - \left(\hat{\vartheta}_{m-1_i}\, P_{m-1}\right)_{Tr} \tag{43}$$

m Cycle Integrations:

$$\hat{\Phi}\left(t,t_{m-1}\right) = I + \int_{t_{m-1}}^{t} \hat{A}\, \hat{\Phi}(t,\tau)\, d\tau \qquad \hat{\underline{v}}\left(t,t_{m-1}\right) = \int_{t_{m-1}}^{t} \hat{\Phi}(t_m,\tau)\, d\tau \tag{44}$$

$$\mathcal{E}\left(\tilde{\underline{x}}(t)\right) = \hat{\Phi}\left(t,t_{m-1}\right) \mathcal{E}\left(\tilde{\underline{x}}_{m-1}\right) + \hat{\underline{v}}\left(t,t_{m-1}\right) \mathcal{E}\left(\tilde{\dot{\underline{x}}}_{2nd_{m-1}}\right) \tag{45}$$

$$\begin{aligned} Q^*_{Dens}(t) = {} & \hat{G}_P(t)\, Q_{P_{Dens}}(t)\, \hat{G}^T_P(t) \\ & + \mathcal{E}\left(\tilde{\dot{\underline{x}}}_{2nd_{m-1}}\right) \mathcal{E}\left(\tilde{\underline{x}}(t)\right)^T + \mathcal{E}\left(\tilde{\underline{x}}(t)\right) \mathcal{E}\left(\tilde{\dot{\underline{x}}}^T_{2nd_{m-1}}\right) \end{aligned} \tag{46}$$

$$Q^*_m = \int_{t_{m-1}}^{t_m} Q^*_{Dens}(t)\, dt \tag{47}$$

Error State Vector And Covariance m Cycle Propagation:

$$\hat{\underline{x}}_m = \hat{\Phi}\left(t_m,t_{m-1}\right) \hat{\underline{x}}_{m-1} + \hat{\underline{v}}\left(t_m,t_{m-1}\right) \hat{\dot{\underline{x}}}_{2nd_{m-1}} \tag{48}$$

$$P_m = \Phi\left(t_m,t_m\right) \left(P_m + \frac{1}{2}\, Q^*_m\right) \Phi^T\left(t_m,t_m\right) + \frac{1}{2}\, Q^*_m \tag{49}$$

When m Corresponds To A Kalman Estimation n Cycle:

Take In Measurement $\underline{z}_n$

$$P_n(-) = P_m \tag{50}$$

$$\hat{\underline{x}}_n(-) = \hat{\underline{x}}_m \tag{51}$$

$$\mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) = \mathcal{E}\left(\tilde{\underline{x}}(t_m)\right) \tag{52}$$

$$\mathcal{E}\left(\Delta\tilde{\underline{z}}_{2nd_{i/n}}(-)\right) = \left(\hat{\Gamma}_{i_n} P_n(-)\right)_{Tr} \tag{53}$$

$$\hat{H}^*_{n_i} \equiv \hat{H}_{n_i} + \hat{\underline{x}}^T_n(-) \left(\hat{\Gamma}_{n_i} + \hat{\Gamma}^T_{n_i}\right) \tag{54}$$

$$R_{M_n} \equiv \hat{G}_{M_n} \mathcal{E}\left(\underline{n}_{M_n} \underline{n}^T_{M_n}\right) \hat{G}^T_{M_n}$$

(55

$$R^*_{M_n} \equiv R_{M_n} + \mathcal{E}\left(\Delta\tilde{\underline{z}}_{2nd_n}(-)\right) \mathcal{E}\left(\Delta\tilde{\underline{z}}^T_{2nd_n}(-)\right)$$

$$\hat{z}_{2nd_{n/i}}(-) = \hat{\underline{x}}^T_n(-)\, \hat{\Gamma}_{n_i}\, \hat{\underline{x}}_n(-) \quad \text{or} \quad \hat{z}_{2nd_{n/i}}(-) = \left(\hat{\Gamma}_{n_i}\, \hat{\underline{x}}_n(-)\, \hat{\underline{x}}^T_n(-)\right)_{Tr} \tag{56}$$

$$K_n = \left[\begin{array}{c} P_n(-)\, \hat{H}^{*T}_n \\ - \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta\tilde{\underline{z}}^T_{2nd_n}(-)\right) \end{array}\right] \left[\begin{array}{c} \hat{H}^*_n P_n(-)\, \hat{H}^{*T}_n + R^*_{M_n} \\ - \hat{H}^*_n\ \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta\tilde{\underline{z}}^T_{2nd_n}(-)\right) \\ - \left[\hat{H}^*_n\ \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta\tilde{\underline{z}}^T_{2nd_n}(-)\right)\right]^T \end{array}\right]^{-1} \tag{57}$$

$$\hat{\underline{z}}_n = \hat{H}_n\, \hat{\underline{x}}_n(-) + \hat{\underline{z}}_{2nd_n}(-) \tag{58}$$

$$\hat{\underline{x}}_n(+) = \hat{\underline{x}}_n(-) + K_n \left(\underline{z}_n - \hat{\underline{z}}_n\right) \tag{59}$$

$$\mathcal{E}\left(\tilde{\underline{x}}_n(+)\right) = \left(I - K_n\, \hat{H}^*_n\right) \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) + K_n\, \mathcal{E}\left(\Delta\tilde{\underline{z}}_{2nd_n}(-)\right) \tag{60}$$

$$\begin{aligned} P_n(+) = P_n(-) + K_n & \left[\begin{array}{c} \hat{H}^*_n P_n(-)\, \hat{H}^{*T}_n + R^*_{M_n} \\ - \hat{H}^*_n\ \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta\tilde{\underline{z}}^T_{2nd_n}(-)\right) \\ - \left[\hat{H}^*_n\ \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta\tilde{\underline{z}}^T_{2nd_n}(-)\right)\right]^T \end{array}\right] K^T_n \\ & - K_n \left[P_n(-)\, \hat{H}^{*T}_n - \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta\tilde{\underline{z}}^T_{2nd_n}(-)\right)\right]^T \\ & - \left[P_n(-)\, \hat{H}^{*T}_n - \mathcal{E}\left(\tilde{\underline{x}}_n(-)\right) \mathcal{E}\left(\Delta\tilde{\underline{z}}^T_{2nd_n}(-)\right)\right] K^T_n \end{aligned} \tag{61}$$

Repeat Above At Kalman n Cycle Update Rate

Note in equations (43), (45), (46), (54), (55), (57), (60), and (61) that as the estimated error state vector uncertainty and its covariance are reduced, the second order augmentation terms in the revised Kalman filter adaptively reduce in magnitude, becoming negligible as the filter converges to a minimum error state uncertainty condition.

SIMPLIFICATIONS

If implemented exactly as shown, equations (44) and (45) would introduce a significant increase in computational requirements compared with the traditional first order Kalman filter approach. By approximating $\hat{\Phi}(t_m,\tau)$ in (44) and $\hat{\Phi}\left(t,t_{m-1}\right)$ in (45) by identity, (45) simplifies to

$$\mathcal{E}\left(\underline{\tilde{x}}(t)\right) \approx \mathcal{E}\left(\underline{\tilde{x}}_{m-1}\right) + (t - t_{m-1})\,\mathcal{E}\left(\underline{\dot{\tilde{x}}}_{2nd_{m-1}}\right)$$

so that for (45) and (47)

$$\begin{aligned}
&\int_{t_{m-1}}^{t_m} \left[\mathcal{E}\left(\underline{\dot{\tilde{x}}}_{2nd_{m-1}}\right) \mathcal{E}\left(\underline{\tilde{x}}^T_{m-1}\right) + \mathcal{E}\left(\underline{\tilde{x}}_{m-1}\right) \mathcal{E}\left(\underline{\dot{\tilde{x}}}^T_{2nd_{m-1}}\right)\right] dt \\
&= \left[\mathcal{E}\left(\underline{\dot{\tilde{x}}}_{2nd_{m-1}}\right) \mathcal{E}\left(\underline{\tilde{x}}^T_{m-1}\right) + \mathcal{E}\left(\underline{\tilde{x}}_{m-1}\right) \mathcal{E}\left(\underline{\dot{\tilde{x}}}^T_{2nd_{m-1}}\right)\right] (t_m - t_{m-1}) \\
&\qquad + \mathcal{E}\left(\underline{\dot{\tilde{x}}}_{2nd_{m-1}}\right) \mathcal{E}\left(\underline{\dot{\tilde{x}}}^T_{2nd_{m-1}}\right) (t_m - t_{m-1})^2
\end{aligned}$$

with which (45) - (47) become

$$\begin{aligned}
Q_m &= \int_{t_{m-1}}^{t_m} \left(\hat{G}_P(t)\, Q_{P_{Dens}}(t)\, \hat{G}_P^T(t)\right) dt \\
Q_m^* &= Q_m + \mathcal{E}\left(\underline{\dot{\tilde{x}}}_{2nd_{m-1}}\right) \mathcal{E}\left(\underline{\dot{\tilde{x}}}^T_{2nd_{m-1}}\right) (t_m - t_{m-1})^2 \\
&+ \left[\mathcal{E}\left(\underline{\dot{\tilde{x}}}_{2nd_{m-1}}\right) \mathcal{E}\left(\underline{\tilde{x}}^T_{m-1}\right) + \mathcal{E}\left(\underline{\tilde{x}}_{m-1}\right) \mathcal{E}\left(\underline{\dot{\tilde{x}}}^T_{2nd_{m-1}}\right)\right] (t_m - t_{m-1}) \\
\mathcal{E}\left(\underline{\tilde{x}}_m\right) &= \mathcal{E}\left(\underline{\tilde{x}}_{m-1}\right) + (t_m - t_{m-1})\, \mathcal{E}\left(\underline{\dot{\tilde{x}}}_{2nd_{m-1}}\right)
\end{aligned} \qquad (62)$$

Similarly, (48) would become

$$\underline{\hat{x}}_m = \hat{\Phi}\left(t_m,t_{m-1}\right) \underline{\hat{x}}_{m-1} + (t_m - t_{m-1})\, \underline{\dot{\hat{x}}}_{2nd_{m-1}} \qquad (63)$$

With these approximations, $\hat{\Phi}\left(t,t_{m-1}\right)$ would no longer be needed at each time point t, and $\hat{\Phi}\left(t_m,t_{m-1}\right)$ for (63) could be calculated using a more traditional approach, e.g., [3 - (15.1.2.1.1.3-37)]:

$$\Delta\hat{\Phi}_m \equiv \int_{t_{m-1}}^{t_m} \hat{A}(t)\, dt \qquad \hat{\Phi}(t_m,t_{m-1}) \approx e^{\Delta\hat{\Phi}_m}$$

$$e^{\Delta\hat{\Phi}_m} \equiv \Delta\hat{\Phi}_m + \frac{1}{2!}\,\Delta\hat{\Phi}_m^2 + \frac{1}{3!}\,\Delta\hat{\Phi}_m^2 + \cdots \tag{64}$$

Thus, equations (44) - (48) would be replaced by (62) - (64) with only $\Delta\hat{\Phi}_m$ and Q_m requiring integration operations over an m cycle (the same as for a traditional Kalman filter), and (44) - (49) would become:

m Cycle Integrations:

$$Q_m = \int_{t_{m-1}}^{t_m} \left(\hat{G}_P(t)\, Q_{P_{Dens}}(t)\, \hat{G}_P^T(t)\right) dt \tag{65}$$

$$\Delta\hat{\Phi}_m \equiv \int_{t_{m-1}}^{t_m} \hat{A}(t)\, dt \tag{66}$$

Error State Vector And Covariance m Cycle Propagation:

$$\hat{\Phi}(t_m,t_{m-1}) = e^{\Delta\hat{\Phi}_m} \tag{67}$$

$$\hat{\underline{x}}_m = \hat{\Phi}(t_m,t_{m-1})\, \hat{\underline{x}}_{m-1} + (t_m - t_{m-1})\, \dot{\hat{x}}_{2nd_{m-1}} \tag{68}$$

$$\begin{aligned} Q_m^* &= Q_m + \mathcal{E}\left(\dot{\tilde{\underline{x}}}_{2nd_{m-1}}\right) \mathcal{E}\left(\dot{\tilde{\underline{x}}}_{2nd_{m-1}}^T\right) (t_m - t_{m-1})^2 \\ &+ \left[\mathcal{E}\left(\dot{\tilde{\underline{x}}}_{2nd_{m-1}}\right) \mathcal{E}\left(\tilde{\underline{x}}_{m-1}^T\right) + \mathcal{E}\left(\tilde{\underline{x}}_{m-1}\right) \mathcal{E}\left(\dot{\tilde{\underline{x}}}_{2nd_{m-1}}^T\right)\right] (t_m - t_{m-1}) \end{aligned} \tag{69}$$

$$\mathcal{E}\left(\tilde{\underline{x}}_m\right) = \mathcal{E}\left(\tilde{\underline{x}}_{m-1}\right) + (t_m - t_{m-1})\, \mathcal{E}\left(\dot{\tilde{\underline{x}}}_{2nd_{m-1}}\right) \tag{70}$$

$$P_m = \Phi(t_m,t_m)\left(P_m + \frac{1}{2}\, Q_m^*\right) \Phi^T(t_m,t_m) + \frac{1}{2}\, Q_m^* \tag{71}$$

CONTROL RESETS

In traditional Kalman filters it is common to make "control resets" at the Kalman filter cycle times whereby the estimated error state vector values are used as the basis for correcting the states being calculated in the system. Simultaneously for compatibility, the estimated error state vector would be reset by the same correction. [3 - Sect. 15.1.2] describes the control reset process for both an idealized computer that can make required calculations instantaneously, and a method for dealing with finite computation time limitations using "delayed control resets". When using the second order Kalman approach described here, it is important that control reset operations are structured to avoid introducing second order errors in the reset process. The example to follow illustrates how this can be done for a Kalman filter applied to a strapdown INS.

Note that when control resets are used, they are applied following a Kalman update, effectively nullifying $\underline{\widehat{x}}_n(+)$. For a traditional Kalman filter, this translates into nullifying $\underline{\widehat{x}}_n(-)$ for the next Kalman cycle, thereby eliminating the need to propagate $\underline{\widehat{x}}_n(+)$ over the next n cycle (except for real-time computational delay considerations [3 - Sect. 15.1.2]). When considering second order effects, however, $\underline{\widehat{x}}_n(-)$ will not be nullified from the reset because it will build into a second order error from $\dot{\underline{\widehat{x}}}_{2nd_{m-1}}$ propagation in equation (48). However, $\underline{\widehat{x}}_n(-)$ will be second order in magnitude so that the $\widehat{H}^*_{i_n}$ augmentation terms in (54) will become negligible. The result is that (54) can be eliminated, and $\widehat{H}^*_{i_n}$ in (57) can be replaced with the traditional $\widehat{H}_n$ form.

It is recommended that if control resets are being used with a second order Kalman estimator, the [3 - Sect. 15.1.2] process should be applied rigorously to make sure that all second order effects are properly accounted for.

STRAPDOWN INS EXAMPLE

In a strapdown INS, angular orientation (attitude) and velocity are calculated by an integration process using gyro sensed angular rotation rates and accelerometer sensed force accelerations as input. The governing differential equations being integrated are from [3 - Sects. 4.1 - 4.3]:

$$\dot{C}^N_B = C^N_B \left(\underline{\omega}^B_{IB} \times\right) - \left(\underline{\omega}^N_{IN} \times\right) C^N_B \tag{72}$$

$$\dot{\underline{v}}^N = C^N_B \, \underline{a}^B_{SF} + \underline{g}^N_P - \left(\underline{\omega}^N_{IN} + \underline{\omega}^N_{IE}\right) \times \underline{v}^N \tag{73}$$

where

$\left(\underline{V}^A \times\right)$ = Skew symmetric (or cross-product) form of general vector $\underline{V}$ projected on general coordinate frame A axes (superscript), as represented by the square matrix $\begin{bmatrix} 0 & -V_{ZA} & V_{YA} \\ V_{ZA} & 0 & -V_{XA} \\ -V_{YA} & V_{XA} & 0 \end{bmatrix}$ in which V_{XA}, V_{YA}, V_{ZA} are the components of $\underline{V}^A$. The matrix product of $\left(\underline{V}^A \times\right)$ with another A frame vector column matrix equals the cross-product of $\underline{V}$ with the vector in the A frame, i.e., $\left(\underline{V}^A \times\right) \underline{W}^A = \left(\underline{V} \times \underline{W}\right)^A$.

N = Navigation coordinate frame used for velocity determination (typically locally level with Z axis up).

B = Sensor coordinate frame (Body frame) to which the inertial sensors (gyros and accelerometers) are mounted, and that rotates with the vehicle containing the INS.

I = Inertially non-rotating coordinate frame.

E = Earth fixed coordinate frame that rotates at earth's rotation rate relative to the I frame.

C_B^N = Direction cosine matrix that transforms a vector projected (coordinatized) on B frame axes (B subscript), to the same vector but coordinatized on N frame axes (N superscript).

$\underline{v}^N$ = Velocity of the INS relative to the earth, coordinatized along N frame axes (N superscript)

$\underline{\omega}_{IB}^B$ = Angular rate of the B frame relative to the I frame (IB subscript) expressed in B frame (superscript) axes (the angular rate vector measured by gyros aligned with B frame axes).

$\underline{\omega}_{IE}^N$ = Angular rate of the E frame relative to the I frame (IE subscript) expressed in N frame coordinates (N superscript).

$\underline{\omega}_{IN}^N$ = Angular rate of the N frame relative to the I frame (IN subscript) expressed in N frame coordinates (N superscript).

$\underline{a}_{SF}^B$ = Specific force acceleration expressed in B frame (superscript) axes (the acceleration vector measured by strapdown accelerometers).

$\underline{g}_P^N$ = Plumb-bob gravity that equals the sum of earth's gravitational mass attraction plus earth's rotation centripetal acceleration effect. Defined as such because $\underline{g}_P^N$ lies along the direction of a plumb-bob under zero velocity conditions).

The attitude and velocity equations implemented in the INS computer are identical in form to (72) and (73):

$$\dot{\hat{C}}_B^N = \hat{C}_B^N \left(\hat{\underline{\omega}}_{IB}^B \times\right) - \left(\hat{\underline{\omega}}_{IN}^N \times\right) \hat{C}_B^N \tag{74}$$

$$\dot{\hat{\underline{v}}}^N = \hat{C}_B^N \, \hat{\underline{a}}_{SF}^B + \hat{\underline{g}}_P^N - \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N\right) \times \hat{\underline{v}}^N \tag{75}$$

Errors in the $\left(\hat{\ }\right)$ parameters in (74) and (75) are defined as:

$$\delta C_B^N \equiv \hat{C}_B^N - C_B^N \qquad \delta \underline{\omega}_{IB}^B \equiv \underline{\hat{\omega}}_{IB}^B - \underline{\omega}_{IB}^B \qquad \delta \underline{\omega}_{IN}^N \equiv \underline{\hat{\omega}}_{IN}^N - \underline{\omega}_{IN}^N \tag{76}$$

$$\begin{gathered} \delta \underline{v}^N \equiv \underline{\hat{v}}^N - \underline{v}^N \qquad \delta \underline{a}_{SF}^B \equiv \underline{\hat{a}}_{SF}^N - \underline{a}_{SF}^N \\ \delta \underline{\omega}_{IE}^N \equiv \underline{\hat{\omega}}_{IE}^N - \underline{\omega}_{IE}^N \qquad \delta \underline{g}_P^N \equiv \underline{\hat{g}}_P^N - \underline{g}_P^N \end{gathered} \tag{77}$$

where

$\delta(\ )$ = Designation for errors that are small compared with $(\ )$.

Attitude Error Equation

Derivation of the attitude error equation associated with (74) begins with substituting definitions (76) into (72):

$$\begin{aligned} \dot{\hat{C}}_B^N - \delta \dot{C}_B^N &= \left(\hat{C}_B^N - \delta C_B^N\right)\left[\left(\underline{\hat{\omega}}_{IB}^B - \delta \underline{\omega}_{IB}^B\right)\times\right] - \left[\left(\underline{\hat{\omega}}_{IN}^N - \delta \underline{\omega}_{IN}^N\right)\times\right]\left(\hat{C}_B^N - \delta C_B^N\right) \\ &= \hat{C}_B^N \left(\underline{\hat{\omega}}_{IB}^B \times\right) - \hat{C}_B^N \left(\delta \underline{\omega}_{IB}^B \times\right) - \delta C_B^N \left(\underline{\tilde{\omega}}_{IB}^B \times\right) + \delta C_B^N \left(\delta \underline{\omega}_{IB}^B \times\right) \\ &\quad - \left(\underline{\hat{\omega}}_{IN}^N \times\right) \hat{C}_B^N + \left(\underline{\hat{\omega}}_{IN}^N \times\right) \delta C_B^N + \left(\delta \underline{\omega}_{IN}^N \times\right) \hat{C}_B^N - \left(\delta \underline{\omega}_{IN}^N \times\right) \delta C_B^N \end{aligned} \tag{78}$$

Substituting (74) for $\dot{\hat{C}}_B^N$ in (78):

$$\begin{aligned} \delta \dot{C}_B^N &= \hat{C}_B^N \left(\delta \underline{\omega}_{IB}^B \times\right) + \delta C_B^N \left(\underline{\tilde{\omega}}_{IB}^B \times\right) - \delta C_B^N \left(\delta \underline{\omega}_{IB}^B \times\right) \\ &\quad - \left(\underline{\hat{\omega}}_{IN}^N \times\right) \delta C_B^N - \left(\delta \underline{\omega}_{IN}^N \times\right) \hat{C}_B^N + \left(\delta \underline{\omega}_{IN}^N \times\right) \delta C_B^N \end{aligned} \tag{79}$$

The δC_B^N error is from (76):

$$\delta C_B^N \equiv \hat{C}_B^N - C_B^N = \left[I - C_B^N \left(\hat{C}_B^N\right)^T \right] \hat{C}_B^N \tag{80}$$

where it has been assumed that the INS software is sufficiently accurate that computed direction cosine matrices satisfy orthogonality/normality constraints, hence, $\left(\hat{C}_B^N\right)^{-1} = \left(\hat{C}_B^N\right)^T$. Applying [3 - (3.5.2-8)] assigns the cause for the C_B^N error to misalignment of the N frame. Identifying the misaligned N frame as $\hat{N}$ gives:

$$\hat{C}_B^N = C_B^{\hat{N}} = C_N^{\hat{N}} \, C_B^N \qquad \text{or} \qquad C_B^N = C_{\hat{N}}^N \, \hat{C}_B^N$$

hence, from (80),

$$\delta C_B^N = \left(I - C_{\hat{N}}^N \right) \hat{C}_B^N \tag{81}$$

Defining $C_{\hat{N}}^N$ in terms of a rotation vector, the [3 - (19.1.3-3) & (19.1.3-7)] form can be used as a model:

$$C_{\hat{N}}^N = I + f_1(\gamma) \left(\underline{\gamma}^N \times \right) + f_2(\gamma) \left(\underline{\gamma}^N \times \right) \left(\underline{\gamma}^N \times \right)$$

$$f_1(\gamma) = \frac{\sin \gamma}{\gamma} = 1 - \frac{\gamma^2}{3\,!} + \cdots \qquad f_2(\gamma) = \frac{\left(1 - \cos \gamma\right)}{\gamma^2} = \frac{1}{2} - \frac{\gamma^2}{4\,!} + \cdots \tag{82}$$

where

$\underline{\gamma}^N$ = Rotation angle error vector associated with the $\hat{C}_B^N$ matrix considering the N frame to be misaligned, as projected on frame N axes.

Substituting (82) in (81) then finds with no approximations:

$$\delta C_B^N = - \left[f_1 \left(\underline{\gamma}^N \times \right) + f_2 \left(\underline{\gamma}^N \times \right) \left(\underline{\gamma}^N \times \right) \right] \hat{C}_B^N \tag{83}$$

and

$$\begin{aligned}
\delta \dot{C}_B^N &= - \left\{ \begin{array}{c} f_1 \left(\dot{\underline{\gamma}}^N \times \right) + \dot{f}_1 \left(\underline{\gamma}^N \times \right) \\ + f_2 \frac{d}{dt} \left[\left(\underline{\gamma}^N \times \right) \left(\underline{\gamma}^N \times \right) \right] + \dot{f}_2 \left(\underline{\gamma}^N \times \right) \left(\underline{\gamma}^N \times \right) \end{array} \right\} \hat{C}_B^N \\
&\quad - \left[f_1 \left(\underline{\gamma}^N \times \right) + f_2 \left(\underline{\gamma}^N \times \right) \left(\underline{\gamma}^N \times \right) \right] \dot{\hat{C}}_B^N \\
&= - \left\{ \begin{array}{c} f_1 \left(\dot{\underline{\gamma}}^N \times \right) + \dot{f}_1 \left(\underline{\gamma}^N \times \right) \\ + f_2 \frac{d}{dt} \left[\left(\underline{\gamma}^N \times \right) \left(\underline{\gamma}^N \times \right) \right] + \dot{f}_2 \left(\underline{\gamma}^N \times \right) \left(\underline{\gamma}^N \times \right) \end{array} \right\} \hat{C}_B^N \\
&\quad - \left[f_1 \left(\underline{\gamma}^N \times \right) + f_2 \left(\underline{\gamma}^N \times \right) \left(\underline{\gamma}^N \times \right) \right] \left[\hat{C}_B^N \left(\hat{\underline{\omega}}_{IB}^B \times \right) - \left(\hat{\underline{\omega}}_{IN}^N \times \right) \hat{C}_B^N \right]
\end{aligned} \tag{84}$$

Then substituting (83) and (84) in (79) obtains after rearrangement:

$$
\begin{aligned}
& f_1 \left(\dot{\underline{\gamma}}^N \times\right) + \dot{f}_1 \left(\underline{\gamma}^N \times\right) \\
& + f_2 \frac{d}{dt}\left[\left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] + \dot{f}_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right) \\
& - \left[f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right]\left(\hat{\underline{\omega}}_{IN}^N \times\right) \\
= & - \left[\left(\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right)\times\right] - \left[f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right]\left[\left(\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right)\times\right] \\
& - \left(\hat{\underline{\omega}}_{IN}^N \times\right)\left[f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] + \left(\delta\underline{\omega}_{IN}^N \times\right) \\
& + \left(\delta\underline{\omega}_{IN}^N \times\right)\left[f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right]
\end{aligned}
\qquad (85)
$$

From its definition, the cross-product matrix operator form of a vector is anti-symmetric (i.e., the element in row i column j equals the negative of the element in row j column i). It follows that the transpose of a cross-product operator matrix equals the negative of the matrix. Based on this property, the transpose of (85) is:

$$
\begin{aligned}
& - f_1 \left(\dot{\underline{\gamma}}^N \times\right) - \dot{f}_1 \left(\underline{\gamma}^N \times\right) \\
& + f_2 \frac{d}{dt}\left[\left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] + \dot{f}_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right) \\
& + \left(\hat{\underline{\omega}}_{IN}^N \times\right)\left[- f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] \\
= & \left[\left(\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right)\times\right] + \left[\left(\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right)\times\right]\left[- f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right] \\
& + \left[- f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right]\left(\hat{\underline{\omega}}_{IN}^N \times\right) - \left(\delta\underline{\omega}_{IN}^N \times\right) \\
& - \left[- f_1 \left(\underline{\gamma}^N \times\right) + f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\right]\left(\delta\underline{\omega}_{IN}^N \times\right)
\end{aligned}
\qquad (86)
$$

Subtracting (86) from (85), dividing by 2, and rearranging obtains:

$$
\begin{aligned}
& f_1 \left(\dot{\underline{\gamma}}^N \times\right) = - \left[\left(\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right)\times\right] + \left(\delta\underline{\omega}_{IN}^N \times\right) \\
& + f_1 \left[\left(- \hat{\underline{\omega}}_{IN}^N + \frac{1}{2}\,\delta\underline{\omega}_{IN}^N + \frac{1}{2}\,\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right)\times\right]\left(\underline{\gamma}^N \times\right) \\
& - f_1 \left(\underline{\gamma}^N \times\right)\left[\left(- \hat{\underline{\omega}}_{IN}^N + \frac{1}{2}\,\delta\underline{\omega}_{IN}^N + \frac{1}{2}\,\hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right)\times\right] \\
& + \frac{1}{2} f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\left[\left(\delta\underline{\omega}_{IN}^N - \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right)\times\right] \\
& + \frac{1}{2} f_2 \left[\left(\delta\underline{\omega}_{IN}^N - \hat{C}_B^N \, \delta\underline{\omega}_{IB}^B\right)\times\right]\left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right) - \dot{f}_1 \left(\underline{\gamma}^N \times\right)
\end{aligned}
$$

or, after applying [3 - (3.1.1-22)]

$$
\begin{aligned}
f_1\left(\dot{\underline{\gamma}}^N \times\right) &= -\left[\left(\hat{C}_B^N\, \delta\underline{\omega}_{IB}^B\right)\times\right] + \left(\delta\underline{\omega}_{IN}^N \times\right) \\
&+ f_1 \left\{\left[\left(-\,\hat{\underline{\omega}}_{IN}^N + \frac{1}{2}\,\delta\underline{\omega}_{IN}^N + \frac{1}{2}\,\hat{C}_B^N\, \delta\underline{\omega}_{IB}^B\right)\times \underline{\gamma}^N\right]\times\right\} \\
&+ \frac{1}{2}\, f_2 \left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right)\left[\left(\delta\underline{\omega}_{IN}^N - \hat{C}_B^N\, \delta\underline{\omega}_{IB}^B\right)\times\right] \\
&+ \frac{1}{2}\, f_2 \left[\left(\delta\underline{\omega}_{IN}^N - \hat{C}_B^N\, \delta\underline{\omega}_{IB}^B\right)\times\right]\left(\underline{\gamma}^N \times\right)\left(\underline{\gamma}^N \times\right) - \dot{f}_1 \left(\underline{\gamma}^N \times\right)
\end{aligned}
\tag{87}
$$

But from (83) and (87),

$$
f_1 = 1 - \frac{\gamma^2}{3\,!} + \cdots = \text{order of } \gamma^2 \qquad \dot{f}_1 = -\,\frac{\gamma}{3}\,\dot{\gamma} + \cdots = \text{order of } \gamma\, \delta\omega_{IN}
$$

hence,

$$
\begin{aligned}
\left(\dot{\underline{\gamma}}^N \times\right) &= -\left[\left(\hat{C}_B^N\, \delta\underline{\omega}_{IB}^B\right)\times\right] + \left(\delta\underline{\omega}_{IN}^N \times\right) \\
&+ \left\{\left[\left(-\,\hat{\underline{\omega}}_{IN}^N + \frac{1}{2}\,\delta\underline{\omega}_{IN}^N + \frac{1}{2}\,\hat{C}_B^N\, \delta\underline{\omega}_{IB}^B\right)\times \underline{\gamma}^N\right]\times\right\} \\
&+ \text{order of } \gamma^2\, \delta\omega_{IN} + \text{order of } \gamma^2\, \delta\omega_{IB}
\end{aligned}
$$

Therefore:

$$
\dot{\underline{\gamma}}^N = -\,\hat{C}_B^N\, \delta\underline{\omega}_{IB}^B - \left(\hat{\underline{\omega}}_{IN}^N - \frac{1}{2}\,\delta\underline{\omega}_{IN}^N - \frac{1}{2}\,\hat{C}_B^N\, \delta\underline{\omega}_{IB}^B\right)\times \underline{\gamma}^N + \delta\underline{\omega}_{IN}^N + \cdots
$$

or

$$
\dot{\underline{\gamma}}^N \approx -\,\hat{C}_B^N\, \delta\underline{\omega}_{IB}^B - \hat{\underline{\omega}}_{IN}^N \times \underline{\gamma}^N + \delta\underline{\omega}_{IN}^N + \frac{1}{2}\left(\hat{C}_B^N\, \delta\underline{\omega}_{IB}^B + \delta\underline{\omega}_{IN}^N\right)\times \underline{\gamma}^N
\tag{88}
$$

Equation (88) would constitute one of the components of the (4) error state dynamic equations. The $\underline{\gamma}^N$ term would form part of the error state vector $\underline{x}$. The $\delta\underline{\omega}_{IB}^B$ error term in (88) would typically be modeled as a sum of several error effects contributing to gyro error, each having its own error state dynamic equation portion of $\dot{\underline{x}}$. The $\delta\underline{\omega}_{IN}^N$ term in (88) is actually a function of the velocity error $\delta\underline{v}^N$ (another component of $\underline{x}$) and position error [3 - (12.3.4-11), -13), & -15)] the latter derived through an integration process on velocity (not included here).

Equation (88) is the error rate equation associated with (74), the equation integrated for attitude updating in the INS computer. Note that (88) is a function of error terms and parameters containing errors, the latter available in the INS computer (the same form as in general error state dynamic equation (4)). This contrasts with the more typical error equations, e.g., [3 - Sects. 12.5.1 - 12.5.4], in which the navigation parameters are represented by their ideal values. Equation (88) is in the form that would actually be used in the INS for a Kalman filter designed to estimate the error parameters. This form arises because of the derivation process followed from (78) in which the ideal attitude updating equation (72) was modified to be a function of computed minus error parameters.

Velocity Error Equation

Derivation of the velocity error equation associated with (75) begins with substituting definitions (77) and (76) into (73) and expanding:

$$
\begin{aligned}
&\dot{\hat{\underline{v}}}^N - \delta\dot{\underline{v}}^N = \left(\hat{C}_B^N - \delta C_B^N\right)\left(\hat{\underline{a}}_{SF}^B - \delta\underline{a}_{SF}^B\right) \\
&+ \hat{\underline{g}}_P^N - \delta\underline{g}_P^N - \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N - \delta\underline{\omega}_{IN}^N - \delta\underline{\omega}_{IE}^N\right) \times \left(\hat{\underline{v}}^N - \delta\underline{v}^N\right) \\
&= \hat{C}_B^N \hat{\underline{a}}_{SF}^B - \delta C_B^N \hat{\underline{a}}_{SF}^B - \left(\hat{C}_B^N - \delta C_B^N\right) \delta\underline{a}_{SF}^B + \hat{\underline{g}}_P^N - \delta\underline{g}_P^N \\
&- \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N\right) \times \hat{\underline{v}}^N + \left(\delta\underline{\omega}_{IN}^N + \delta\underline{\omega}_{IE}^N\right) \times \hat{\underline{v}}^N \\
&+ \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N\right) \times \delta\underline{v}^N - \left(\delta\underline{\omega}_{IN}^N + \delta\underline{\omega}_{IE}^N\right) \times \delta\underline{v}^N
\end{aligned}
$$

or with (75)

$$
\begin{aligned}
\delta\dot{\underline{v}}^N &= \delta C_B^N \hat{\underline{a}}_{SF}^B + \hat{C}_B^N \delta\underline{a}_{SF}^B + \delta\underline{g}_P^N \\
&- \left(\delta\underline{\omega}_{IN}^N + \delta\underline{\omega}_{IE}^N\right) \times \hat{\underline{v}}^N - \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N\right) \times \delta\underline{v}^N \\
&- \delta C_B^N \delta\underline{a}_{SF}^B + \left(\delta\underline{\omega}_{IN}^N + \delta\underline{\omega}_{IE}^N\right) \times \delta\underline{v}^N
\end{aligned}
\qquad (89)
$$

Substituting (83) for δC_B^N in (89) finds with no approximations:

$$
\begin{aligned}
\delta\dot{\underline{v}}^N &= \hat{C}_B^N \delta\underline{a}_{SF}^B + f_1 \left(\hat{C}_B^N \hat{\underline{a}}_{SF}^B\right) \times \underline{\gamma}^N + \delta\underline{g}_P^N \\
&- \left(\delta\underline{\omega}_{IN}^N + \delta\underline{\omega}_{IE}^N\right) \times \hat{\underline{v}}^N - \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N\right) \times \delta\underline{v}^N \\
&- f_2 \left[\left(\hat{C}_B^N \hat{\underline{a}}_{SF}^B\right) \times \underline{\gamma}^N\right] \times \underline{\gamma}^N - f_1 \left(\hat{C}_B^N \delta\underline{a}_{SF}^B\right) \times \underline{\gamma}^N \\
&+ \left(\delta\underline{\omega}_{IN}^N + \delta\underline{\omega}_{IE}^N\right) \times \delta\underline{v}^N + f_2 \left[\left(\hat{C}_B^N \delta\underline{a}_{SF}^B\right) \times \underline{\gamma}^N\right] \times \underline{\gamma}^N
\end{aligned}
\qquad (90)
$$

Then substituting f_1 and f_2 from (82) and neglecting terms with third order error products, (90) becomes:

$$
\begin{aligned}
\delta\dot{\underline{v}}^N \approx{}& \hat{C}_B^N\, \delta\underline{a}_{SF}^B + \hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N + \delta\underline{g}_P^N \\
& - \left(\delta\underline{\omega}_{IN}^N + \delta\underline{\omega}_{IE}^N\right) \times \hat{\underline{v}}^N - \left(\hat{\underline{\omega}}_{IN}^N + \hat{\underline{\omega}}_{IE}^N\right) \times \delta\underline{v}^N \\
& - \frac{1}{2}\left(\hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N\right) \times \underline{\gamma}^N - \left(\hat{C}_B^N\, \delta\underline{a}_{SF}^B\right) \times \underline{\gamma}^N + \left(\delta\underline{\omega}_{IN}^N + \delta\underline{\omega}_{IE}^N\right) \times \delta\underline{v}^N
\end{aligned}
\tag{91}
$$

Equation (91) would constitute one of the components of the (4) error state dynamic equations. The $\delta\underline{v}^N$ term would form part of the error state vector $\underline{x}$. The $\delta\underline{a}_{SF}^B$ error term in (91) would typically be modeled as a sum of several error effects contributing to accelerometer error, each having its own error state dynamic equation portion of $\underline{x}$. The $\delta\underline{\omega}_{IE}^N$ and $\delta\underline{g}_P^N$ terms in (91) are actually functions of the position error [3 - (12.3.4-13) & Sect. 12.2.4], the latter derived through an integration process on velocity (not included here).

Impact On A Velocity Type Measurement

A common measurement for an INS is the comparison between INS velocity and another input reference velocity (i.e., so-called "velocity matching"). Variations exist depending on the particular application (e.g., integrated velocity difference, position comparison), however, each in some form would be based on the velocity error $\delta\underline{v}^N$ (obtained from the integral of (91)).

Consider, for example, when the measurement is formed directly from the difference between the INS computed velocity $\hat{\underline{v}}^N$ and a reference velocity $\hat{\underline{v}}_{REF}^N$. The difference $\hat{\underline{v}}^N - \hat{\underline{v}}_{REF}^N$ cancels the true velocity component $\underline{v}^N$ in each, generating the measurement $\underline{z}^N$ as a function of the errors (uncertainties) in each:

$$
\underline{z}^N = \hat{\underline{v}}^N - \hat{\underline{v}}_{REF}^N = \delta\underline{v}^N - \delta\underline{v}_{REF}^N \tag{92}
$$

The $\delta\underline{v}^N$ INS velocity error term in (92) is the integral of (91) whose pertinent terms for this discussion are with [3 - (3.1.1-16)]:

$$
\begin{aligned}
\delta\dot{\underline{v}}^N &= \hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N - \frac{1}{2}\left(\hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N\right) \times \underline{\gamma}^N + \cdots \\
&= \hat{\underline{a}}_{SF}^N \times \underline{\gamma}^N + \frac{1}{2}\gamma^2\, \hat{\underline{a}}_{SF}^N - \frac{1}{2}\left(\hat{\underline{a}}_{SF}^N \cdot \underline{\gamma}^N\right) \underline{\gamma}^N + \cdots
\end{aligned}
\tag{93}
$$

The $\underline{\gamma}^N$ term in (93) is the integral of INS attitude error equation (88). Over a short time interval t, (88) can be approximated by $\dot{\underline{\gamma}}^N = -\,\hat{\underline{\omega}}_{IN}^N \times \underline{\gamma}_0^N + \cdots$ whose integral is

$$\underline{\gamma}^N = \underline{\gamma}_0^N - \hat{\underline{\omega}}_{IN}^N \times \underline{\gamma}_0^N \, t + \cdots \tag{94}$$

where $\underline{\gamma}_0^N$ is the initial value of $\underline{\gamma}^N$ at t = 0. Substituting (94) in (93), employing the vector triple cross-product identity [3 - (3.1.1-16)], and dropping the contribution of $\hat{\underline{\omega}}_{IN}^N \times \underline{\gamma}_0^N \, t$ in the second order term as negligible, then gives

$$\begin{aligned}
\delta\dot{\underline{v}}^N &= \hat{\underline{a}}_{SF}^N \times \left(\underline{\gamma}_0^N - \hat{\underline{\omega}}_{IN}^N \times \underline{\gamma}_0^N \, t\right) + \frac{1}{2}\gamma_0^2 \, \hat{\underline{a}}_{SF}^N \\
&\quad - \frac{1}{2}\left[\hat{\underline{a}}_{SF}^N \cdot \left(\underline{\gamma}_0^N - \hat{\underline{\omega}}_{IN}^N \times \underline{\gamma}_0^N \, t\right)\right]\left(\underline{\gamma}_0^N - \hat{\underline{\omega}}_{IN}^N \times \underline{\gamma}_0^N \, t\right) + \cdots \\
&\approx \hat{\underline{a}}_{SF}^N \times \underline{\gamma}_0^N - \left(\hat{\underline{a}}_{SF}^N \cdot \underline{\gamma}_0^N\right)\hat{\underline{\omega}}_{IN}^N \, t + \left(\hat{\underline{a}}_{SF}^N \cdot \hat{\underline{\omega}}_{IN}^N \, t\right)\underline{\gamma}_0^N \\
&\qquad + \frac{1}{2}\gamma_0^2 \, \hat{\underline{a}}_{SF}^N - \frac{1}{2}\left(\hat{\underline{a}}_{SF}^N \cdot \underline{\gamma}_0^N\right)\underline{\gamma}_0^N + \cdots
\end{aligned} \tag{95}$$

Equating $\hat{\underline{\gamma}}_0^N$ and $\hat{\underline{a}}_{SF}^N$ to the sum of their horizontal (H) and vertical components:

$$\hat{\underline{\gamma}}_0^N = \underline{\gamma}_{0_H}^N + \gamma_{0_Z}^N \, \underline{u}_Z^N \qquad \hat{\underline{a}}_{SF}^N = \hat{\underline{a}}_{SF_H}^N + \hat{a}_{SF_Z}^N \, \underline{u}_Z^N \tag{96}$$

where

$\underline{u}_Z^N$ = Unit vector along the Z axis of the N frame (vertical up),

the horizontal (H) component of (95) then becomes:

$$\begin{aligned}
\delta\dot{\underline{v}}_H^N &= a_{SF_Z}^N \, \underline{u}_Z^N \times \underline{\gamma}_{0_H}^N + \left(\hat{\underline{a}}_{SF_H}^N \times \underline{u}_Z^N\right)\gamma_{0_Z}^N \\
&\quad - \left(\hat{\underline{a}}_{SF}^N \cdot \underline{\gamma}_0^N\right)\hat{\underline{\omega}}_{IN_H}^N \, t + \left(\hat{\underline{a}}_{SF}^N \cdot \hat{\underline{\omega}}_{IN}^N \, t\right)\underline{\gamma}_{0_H}^N \\
&\quad + \frac{1}{2}\gamma_0^2 \, \hat{\underline{a}}_{SF_H}^N - \frac{1}{2}\left(\hat{\underline{a}}_{SF}^N \cdot \underline{\gamma}_0^N\right)\underline{\gamma}_{0_H}^N + \cdots \\
&= \left[a_{SF_Z}^N \left(\underline{u}_Z^N \times\right) + I \, \hat{\underline{a}}_{SF}^N \cdot \hat{\underline{\omega}}_{IN}^N \, t\right]\underline{\gamma}_{0_H}^N - \left(\hat{\underline{a}}_{SF_H}^N \cdot \underline{\gamma}_{0_H}^N\right)\hat{\underline{\omega}}_{IN_H}^N \, t \\
&\quad - \hat{a}_{SF_Z}^N \left(\hat{\underline{\omega}}_{IN_H}^N \, t\right)\gamma_{0_Z}^N + \left(\hat{\underline{a}}_{SF_H}^N \times \underline{u}_Z^N\right)\gamma_{0_Z}^N \\
&\quad + \frac{1}{2}\gamma_0^2 \, \hat{\underline{a}}_{SF_H}^N - \frac{1}{2}\left(\hat{\underline{a}}_{SF_H}^N \cdot \underline{\gamma}_{0_H}^N + \hat{a}_{SF_Z}^N \, \gamma_{0_Z}^N\right)\underline{\gamma}_{0_H}^N + \cdots
\end{aligned} \tag{97}$$

Substituting $\gamma_0^2 = \underline{\gamma}_{0_H}^N \cdot \underline{\gamma}_{0_H}^N + \left(\gamma_{0_Z}^N\right)^2$ into $\frac{1}{2}\gamma_0^2 \, \hat{\underline{a}}_{SF_H}^N$, combining with $\frac{1}{2}\left(\hat{\underline{a}}_{SF_H}^N \cdot \underline{\gamma}_{0_H}^N\right)\underline{\gamma}_{0_H}^N$ in the third row of (97), and applying the vector triple cross-product identity obtains the final result:

$$
\begin{aligned}
\delta \dot{\underline{v}}^N_H = &\left[\hat{a}^N_{SF_Z} \left(\underline{u}^N_Z \times \right) + I \; \hat{\underline{a}}^N_{SF} \cdot \left(\hat{\underline{\omega}}^N_{IN} \, t \right) \right] \underline{\gamma}^N_{0_H} - \left(\hat{\underline{\omega}}^N_{IN_H} \, t \right) \left(\hat{\underline{a}}^N_{SF_H} \cdot \underline{\gamma}^N_{0_H} \right) \\
& - \hat{a}^N_{SF_Z} \left(\hat{\underline{\omega}}^N_{IN_H} \, t \right) \gamma^N_{0_Z} + \left(\hat{\underline{a}}^N_{SF_H} \times \underline{u}^N_Z \right) \gamma^N_{0_Z} \\
& - \frac{1}{2} \left(\hat{\underline{a}}^N_{SF_H} \times \underline{\gamma}^N_{0_H} \right) \times \underline{\gamma}^N_{0_H} - \frac{1}{2} \hat{a}^N_{SF_Z} \, \underline{\gamma}^N_{0_H} \, \gamma^N_{0_Z} + \frac{1}{2} \hat{\underline{a}}^N_{SF_H} \, {\gamma^N_{0_Z}}^2 + \cdots
\end{aligned}
\qquad (98)
$$

In analyzing the significance of (98) next, it is important to recognize that $\hat{a}^N_{SF_Z}$ consists of a vertical upward component to counteract gravity plus any additional vertical acceleration that would change the vertical velocity relative to the earth. Thus, for predominantly horizontal maneuvering or for small total maneuvering, $\hat{a}^N_{SF_Z}$ would be approximately one g upward.

The integral of (98) is the basic INS error input to the velocity type measurement being considered. The first two lines of (98) are what would have been obtained by neglecting second order effects (all in the third line). The $\hat{a}^N_{SF_Z} \left(\hat{\underline{\omega}}^N_{IN_H} \, t \right)$ coupling in the second line is what makes $\gamma^N_{0_Z}$ observable under non-maneuvering horizontal motion (when $\hat{\underline{a}}^N_{SF_H}$ is zero). Under horizontal motion, the coupling of $\gamma^N_{0_Z}$ by $\hat{\underline{a}}^N_{SF_H} \times \underline{u}^N_Z$ in the second line would be much stronger than $\hat{a}^N_{SF_Z} \left(\hat{\underline{\omega}}^N_{IN_H} \, t \right)$ due to the comparative smallness of $\hat{a}^N_{SF_Z} \left(\hat{\underline{\omega}}^N_{IN_H} \, t \right)$ (particularly for small t). Note also that under horizontal acceleration, the $\hat{\underline{a}}^N_{SF_H} \times \underline{u}^N_Z$ term in the second line dominates the second order $\hat{a}^N_{SF_Z} \, \underline{\gamma}^N_{0_H}$ term in the third line (that would normally be neglected under traditional linear Kalman filter design practice). Under small horizontal acceleration, however, the $\hat{a}^N_{SF_Z} \, \underline{\gamma}^N_{0_H}$ second order term could also dominate the $\hat{\underline{a}}^N_{SF_H}$ term in the third line, and under small t, would dominate the $\hat{a}^N_{SF_Z} \left(\hat{\underline{\omega}}^N_{IN_H} \, t \right)$ term in the second line as well (i.e., during the initial phases of Kalman filter estimation). Unless the second order $\hat{a}^N_{SF_Z} \, \underline{\gamma}^N_{0_H}$ term is accounted for in the Kalman filter design, the result could be a mis-estimation of $\gamma^N_{0_Z}$ heading error. This is what was experienced at SAI under simulated stationary Kalman filter validation testing operations (discussed in the Introduction) which eventually led to preparation of this article.

Applying Control Resets

When correcting state parameter errors using control resets, it is important that higher order errors are not introduced, particularly when attempting to mitigate second order errors in the Kalman design/estimation process. Reference [3 - Sect. 15.1.2.3] illustrates an exact reset method for the INS example being considered in which the state parameters are corrected by subtracting the corresponding estimated error states. For angular orientation states represented by a direction cosine matrix, the exact correction for estimated angular error state vector $\underline{\hat{\gamma}}^N$ components would be:

$$\hat{C}_B^N(+) = \left[I + \frac{\sin \hat{\gamma}}{\hat{\gamma}} \left(\underline{\hat{\gamma}}^N \times \right) + \frac{(1 - \cos \hat{\gamma})}{\hat{\gamma}^2} \left(\underline{\hat{\gamma}}^N \times \right)^2 \right] \hat{C}_B^N(-) \qquad (99)$$

where (-) and (+) refer to the errors states immediately before and after the direction cosine state parameter reset. For a velocity control reset, the method is more straight forward and obvious:

$$\underline{\hat{v}}^N(+) = \underline{\hat{v}}^N(-) - \delta\underline{\hat{v}}^N \qquad (100)$$

For the time instant when resets are being applied, the $\delta\underline{\hat{v}}^N$ and $\underline{\hat{\gamma}}^N$ estimated error state vectors would correspondingly be reset to zero. When "delayed" rather than ideal control resets" are being used, the $\delta\underline{\hat{v}}^N$ and $\underline{\hat{\gamma}}^N$ resets would also account for additional error buildup before the control reset is applied [3 - Sect. 15.1.2].

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