REDEFINING GRAVITY AND NEWTONIAN NATURAL MOTION

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ABSTRACT

Traditional Newtonian mechanics treats gravity as one of the forces contributing to the acceleration of a body in motion. Newtonian natural motion is defined relative to an abstract inertial space where gravity is non-existent and free body velocity motion remains constant unless modified by applied force. With this interpretation, gravitational force is an embedded characteristic whose effect on body acceleration is not measurable by accelerometers, the traditional instruments used in inertial navigation systems to measure acceleration relative to inertial space. This paper introduces a revised interpretation in which gravity is an integral part of natural motion, and natural motion can only be modified by applied non-gravitational forces. With this new interpretation, gravitational force is non-existent, all forces impacting natural motion are measurable by accelerometers, and gravity can only be determined relative to its value at another location. Equations of motion are presented for the new interpretation and used to describe classical known situations: forces experienced on the surface of the earth, "zero-gravity" in free-fall and earth orbit, creating "zero-gravity" in an aircraft, the general relativity principle of equivalence between inertial and gravitational mass, linear and rotational dynamics of mass groups, and a measurable definition of inertial coordinates including its use as an inertial angular reference.

INTRODUCTION

The basic laws of natural motion formulated by Newton in 1667 postulated that in the absence of applied force, a body will translate at a constant natural motion (velocity) [1 - pp. 416]. Velocity is defined as position change relative to an arbitrary non-gravitational inertial space, hence, is a relative rather than absolute quantity. Applied force defined by Newton includes magnetic, electrical, mechanical, and gravitational effects. Under applied force, the body will change its velocity (accelerate) in the direction of the force with magnitude inversely proportional to its inertia (mass) according to Newton's classic equation: $\underline{F} = m_I \, \underline{a}$ where $\underline{F}$ is the applied force vector, m_I is the body inertial mass, and $\underline{a}$ is the associated acceleration vector response.

Newton also postulated that the gravitational force of a body (weight) is proportional to the gravity field in which it is located according to the equation: $\underline{W} = m_G \, \underline{g}$ where $\underline{W}$ is the body weight, m_G is the body gravitational mass, and $\underline{g}$ is the local gravity vector.

Einstein further postulated that an equality between the inertial and gravitational mass (m_I and m_G) is fundamental to the basic laws of general relativity [2 - pp. 78].

This paper introduces a revised formulation of natural motion in which gravity is a fundamental component, and in which deviations from natural motion are produced only by applied measurable forces. As a result, the concepts of gravitational force and gravitational mass are no longer required to describe general motion and physical phenomena. The revised definition of natural motion uses relative acceleration (rather than velocity) as a simpler representation of natural motion, dividing it into two parts; a force acceleration component (measurable with accelerometers), and an immeasurable natural component (of which gravity is an analytically definable constituent). A consequence of the revised natural motion definition is that absolute gravity is an immeasurable quantity, and only relative gravity between two defined locations can be measured by any means.

In the Newtonian formulation, natural body motion is defined as uniform velocity unless modified by applied force, with force including a gravitational component. In the revised formulation there is no gravitational force, and natural motion is "free-fall" velocity as modified by local gravitational acceleration, with gravity being a property of the local body position location in the universe. Deviations from natural motion are generated from forces applied to the body. A consequence of the revised formulation is that the Newtonian concept of "gravitational" mass is not required to predict body motion, nor the general relativity requirement for equivalency between "gravitational" and "inertial" mass.

Another basic difference between the Newtonian and revised formulations is the definition for the coordinate frame used to describe natural motion. Both the Newtonian and revised formulations postulate the existence of an "inertial coordinate frame" in which the laws of motion are valid. In the Newtonian formulation, natural motion is defined for a single body, necessitating that the inertial coordinate frame have a definable origin from which position motion is measured. The inertial frame is defined to be one of constant natural motion velocity in a hypothetical gravity-free space. For the revised approach, motion is defined as the relativistic difference in the movement between two separate masses. As a result, the inertial coordinate frame for describing motion needs no position origin. It is only used as a means for describing the relative angular orientation between different inertial coordinate frames that may be selected to describe relative motion phenomena. The concept of a coordinate frame thereby reduces to an angular orientation definition of three orthogonal unit free-vectors whose mathematical dot products with relative motion vector parameters (e.g., relative position and velocity) define the components of the vector parameters in the coordinate frame. The unit vectors can be conveniently defined in terms of their individual angular orientation relative to definable and observable celestial phenomena (e.g., parallel to a line between two stars, parallel to a perpendicular to the earth-sun ecliptic plane, or parallel to earth's rotation axis).

For both the Newtonian and revised formulations, the "inertial coordinate frame" in which motion is defined includes a requirement for non-rotation. However, the basic concept of rotation implies a changing angular orientation relative to an angular reference. Thus, the concept of non-rotation becomes ambiguous without defining the angular reference from which rotation is to be measured. Such an angular reference has never been defined for the constant velocity translating Newtonian inertial frame formulated in hypothetical gravity free space. Because only the angular orientation characteristics of inertial coordinates is required for the new formulation, the associated unit vector characteristics can be conveniently defined in terms of their individual angular orientation relative to definable and observable celestial phenomena.

This paper defines and provides the analytical basis for the new formulation, deriving the associated laws of translational motion for individual point masses, point mass groups, and rigid bodies; and for the rotational motion of mass groups and rigid bodies. Included is a discussion of the operation of accelerometers used to measure force acceleration for the new formulation, and of gyros used to measure angular rotation relative to the revised definition of inertial coordinates. Examples are provided illustrating how the revised theory explains known phenomena existing in common experience.

NOTATION

$\underline{V}$ = Vector without specific coordinate frame designation. A vector is a parameter that has length and direction. Vectors used in the paper are classified as “free vectors”, hence, have no preferred location in coordinate frames in which they are analytically described.

$\underline{V}^A$ = Column matrix with elements equal to the projection of $\underline{V}$ on coordinate frame A axes. The projection of $\underline{V}$ on each frame A axis equals the dot product of $\underline{V}$ with a unit vector parallel to that coordinate axis.

$\left(\underline{V}^A \times\right)$ = Skew symmetric (or cross-product) form of $\underline{V}^A$ represented by the square matrix $\begin{bmatrix} 0 & -V_{ZA} & V_{YA} \\ V_{ZA} & 0 & -V_{XA} \\ -V_{YA} & V_{XA} & 0 \end{bmatrix}$ in which V_{XA}, V_{YA}, V_{ZA} are the components of $\underline{V}^A$. The matrix product of $\left(\underline{V}^A \times\right)$ with another A frame vector equals the cross-product of $\underline{V}^A$ with the vector in the A frame, i.e.: $\left(\underline{V}^A \times\right) \underline{W}^A = \underline{V}^A \times \underline{W}^A$.

$C_{A_2}^{A_1}$ = Direction cosine matrix that transforms a vector from its coordinate frame A_2 projection form to its coordinate frame A_1 projection form, i.e.:

$\underline{V}^{A_1} = C_{A_2}^{A_1}\, \underline{V}^{A_2}$. The columns of $C_{A_2}^{A_1}$ are projections on A_1 axes of unit vectors parallel to A_2 axes. Conversely, the rows of $C_{A_2}^{A_1}$ are projections on A_2 axes of unit vectors parallel to A_1 axes. An important property of $C_{A_2}^{A_1}$ is that it's inverse equals it's transpose

$\underline{\omega}_{A_1 A_2}$ = Angular rotation rate of coordinate frame A_2 relative to coordinate frame A_1. Conversely, the angular rotation rate of coordinate frame A_1 relative to coordinate frame A_2 is the negative of $\underline{\omega}_{A_1 A_2}$, i.e.,: $\underline{\omega}_{A_2 A_1} = -\underline{\omega}_{A_1 A_2}$

$\dot{(\ )} = \dfrac{d(\ )}{dt}$ = Derivative with respect to time t.

THE DYNAMICS OF POINT MASSES

In this revised formulation, velocity change of a mass point measured in an inertial reference frame is produced by two effects; force applied to the mass (measurable by accelerometers), and non-measurable acceleration that is a property of the position location of the mass in celestial space. Non-measurable acceleration includes gravitational effects and other thus far unknown non-measurable accelerations. Natural motion in the new formulation is motion in the absence of measurable force, hence, accelerated motion produced by local non-measurable acceleration (e.g., gravity) at the position location of the point mass.

From an analytical standpoint, consider two mass points 1 and 2. The relative velocity (position change) between the two points as projected on the axes of an arbitrary coordinate frame A equals the integral of the relative acceleration between the points:

$$\frac{d}{dt}\underline{r}_{2,1}^{A} = \int_0^t \left(\frac{d^2}{dt}\underline{r}_{2,1}^{A} \right) dt + \left(\frac{d}{dt}\underline{r}_{2,1}^{A} \right)_0 \qquad (1)$$

with

$$\underline{r}_{2,1}^{A} = \int_0^t \left(\frac{d}{dt}\underline{r}_{2,1}^{A} \right) dt + \underline{r}_{2,1_0}^{A} \qquad (2)$$

where

$\underline{r}_{2,1}^{A}$ = Column matrix with components of relative distance vector $\underline{r}_{2,1}$ projected along coordinate frame A axes.

$\frac{d}{dt}\underline{r}^A_{2,1}$ = Relative velocity in frame A between mass points 1 and 2.

$\frac{d^2}{dt}\underline{r}^A_{2,1}$ = Relative acceleration in frame A between mass points 1 and 2.

$\underline{r}^A_{2,1_0}, \left(\frac{d}{dt}\underline{r}^A_{2,1}\right)_0$ = Initial values of $\underline{r}^A_{2,1}, \frac{d}{dt}\underline{r}^A_{2,1}$ at time t = 0.

For the revised formulation, if frame A is an inertial (I) frame (to be defined subsequently), the relative acceleration between the mass points equates to the sum of applied force effects and natural motion acceleration:

$$\frac{d^2}{dt}\underline{r}^I_{2,1} = \underline{a}^I_{F_2} - \underline{a}^I_{F_1} + \Delta\underline{a}^I_{N_{2,1}} \qquad \Delta\underline{a}^I_{N_{2,1}} \equiv \underline{a}^I_{N_2} - \underline{a}^I_{N_1} \tag{3}$$

with (1) and (2) specialized for the I frame:

$$\begin{aligned} \frac{d}{dt}\underline{r}^I_{2,1} &= \int_0^t \left(\frac{d^2}{dt}\underline{r}^I_{2,1}\right) dt + \left(\frac{d}{dt}\underline{r}^I_{2,1}\right)_0 \\ \underline{r}^I_{2,1} &= \int_0^t \left(\frac{d}{dt}\underline{r}^I_{2,1}\right) dt + \underline{r}^I_{2,1_0} \end{aligned} \tag{4}$$

where

$\underline{a}_{F_1}, \underline{a}_{F_2}$ = Acceleration of mass points 1 and 2 produced by measurable forces at points 1 and 2.

$\underline{a}_{N_1}, \underline{a}_{N_2}$ = Acceleration of mass points 1 and 2 produced by non-measurable natural properties of the point 1 and 2 position locations.

As in Newton's formulation, the force created accelerations satisfy:

$$\underline{a}^I_{F_1} = F_1 / m_1 \qquad \underline{a}^I_{F_2} = F_2 / m_2 \tag{5}$$

where

m_1, m_2 = Inertial mass of mass points 1 and 2.

F_1, F_2 = Measurable forces applied to mass points 1 and 2.

Equation (3) and (5) with (4) are the fundamental relativistic motion equations for the revised version of natural motion, a revised formulation of Newton's second law and gravitational law [1 - pp. 416 - 417 & pp. 810 - 811]. Equation (3) shows that in the absence of applied force acceleration, the relative velocity between point masses (the integral of (3)) will not be constant, but will change by the difference in natural acceleration at the point mass locations. For the gravitational component of natural motion acceleration at an arbitrary point p in space ($\underline{a}_{N/g_p}$), the classical inverse square Newtonian expression applies, equating gravity to the combined effect of celestial mass:

$$\underline{a}_{N/g_p} = \int \frac{\mu}{r_{p/dm}^3} \underline{r}_{p/dm} \, dm \qquad (6)$$

where

dm = Differential mass element in the universe.

$\underline{r}_{p/dm}$ = Linear distance vector from point p to dm.

$r_{p/dm}$ = Magnitude of $\underline{r}_{p/dm}$.

μ = The universal gravitational constant.

and the integral is over all the mass in the universe.

From (3) we see that although $\underline{a}_{N_2}$ and $\underline{a}_{N_1}$ are not measurable directly (and the gravitational portion can be theoretically calculated from a (6) type analytical model), their relative difference $\Delta\underline{a}^I_{N_{2,1}}$ can still be ascertained from measurements of $\frac{d^2}{dt}\underline{r}^I_{2,1}$ (e.g., optically) and $\underline{a}_{F_1}, \underline{a}_{F_2}$ (using (5) with individual F_1, F_2 force measurements), or from $\frac{d^2}{dt}\underline{r}^I_{2,1}$ alone in the absence of applied force.

It is now stated without explicit proof that based on (3), only relative natural motion acceleration (i.e., the difference between $\underline{a}_{N_2}$ and $\underline{a}_{N_1}$) can ever be measured. Thus, the absolute gravity component of natural motion is not measurable, only its relative effect on the motion between separated bodies. In contrast, $\underline{a}_{F_2}$ and $\underline{a}_{F_1}$ are absolute quantities whose values can be determined at any point in space by individual force measurements.

Appendices A and B expand the results in this section to the translational dynamics of mass groups and rigid bodies. The results are identical to what has been traditionally obtained in the past using the traditional Newtonian formulation, e.g., [3 - Chapts. 3 & 4].

MOTION IN A ROTATING COORDINATE FRAME

Consider a coordinate frame B at an arbitrary angular orientation relative to Inertial frame I. The relative position vector $\underline{r}_{2,1}$ viewed in frame B is related to that in frame I according to

$$\underline{r}_{2,1}^{B} = C_{I}^{B} \, \underline{r}_{2,1}^{I} \tag{7}$$

The relative velocity between points 1 and 2 measured in the B frame (the derivative of $\underline{r}_{2,1}^{B}$) is related to the B frame measured velocity by the derivative of (7):

$$\left(\dot{\underline{r}}_{2,1}^{B}\right) = C_{I}^{B}\left(\dot{\underline{r}}_{2,1}^{I}\right) + \left(\dot{C}_{I}^{B}\right)\underline{r}_{2,1}^{I} \tag{8}$$

It is well known that the time rate of change of a direction cosine matrix is a function of the relative angular rate between it's two relating coordinate frames. In the case of $\left(\dot{C}_{I}^{B}\right)$ in (8), the relationship is [4 - pp. 3-54]:

$$\left(\dot{C}_{I}^{B}\right) = -\left(\underline{\omega}_{IB}^{B}\times\right)C_{I}^{B} \tag{9}$$

where

$\underline{\omega}_{IB}^{B} =$ The angular rate of frame B relative to frame I as measured (superscript) in the B frame.

Substituting (9) in (8) obtains

$$\left(\dot{\underline{r}}_{2,1}^{B}\right) = C_{I}^{B}\left(\dot{\underline{r}}_{2,1}^{I}\right) - \left(\underline{\omega}_{IB}^{B}\times\right)C_{I}^{B}\,\underline{r}_{2,1}^{I} = C_{I}^{B}\left(\dot{\underline{r}}_{2,1}^{I}\right) - \left(\underline{\omega}_{IB}^{B}\times\right)\underline{r}_{2,1}^{B} \tag{10}$$

The relative acceleration between points 1 and 2 in the B frame $\left(\ddot{\underline{r}}_{2,1}^{B}\right)$ is the derivative of (10) which with (9) is

$$\begin{aligned}\left(\ddot{\underline{r}}_{2,1}^{B}\right) &= \left(\dot{C}_{I}^{B}\right)\left(\dot{\underline{r}}_{2,1}^{I}\right) + C_{I}^{B}\left(\ddot{\underline{r}}_{2,1}^{I}\right) - \left(\dot{\underline{\omega}}_{IB}^{B}\times\right)\underline{r}_{2,1}^{B} - \left(\underline{\omega}_{IB}^{B}\times\right)\left(\dot{\underline{r}}_{2,1}^{B}\right) \\ &= -\left(\underline{\omega}_{IB}^{B}\times\right)C_{I}^{B}\left(\dot{\underline{r}}_{2,1}^{I}\right) + C_{I}^{B}\left(\ddot{\underline{r}}_{2,1}^{I}\right) - \left(\dot{\underline{\omega}}_{IB}^{B}\times\right)\underline{r}_{2,1}^{B} - \left(\underline{\omega}_{IB}^{B}\times\right)\left(\dot{\underline{r}}_{2,1}^{B}\right)\end{aligned} \tag{11}$$

Solving for $C_{I}^{B}\left(\dot{\underline{r}}_{2,1}^{I}\right)$ from (8) and substitution in (11) yields

$$\left(\ddot{\underline{r}}_{2,1}^{B}\right) = C_I^B \left(\ddot{\underline{r}}_{2,1}^{I}\right) - \left(\dot{\underline{\omega}}_{IB}^{B} \times\right) \underline{r}_{2,1}^{B} - \left(\underline{\omega}_{IB}^{B} \times\right)\left(\underline{\omega}_{IB}^{B} \times\right) \underline{r}_{2,1}^{B} - 2\left(\underline{\omega}_{IB}^{B} \times\right)\left(\dot{\underline{r}}_{2,1}^{B}\right)$$

$$= C_I^B \left(\ddot{\underline{r}}_{2,1}^{I}\right) - \dot{\underline{\omega}}_{IB}^{B} \times \underline{r}_{2,1}^{B} - \underline{\omega}_{IB}^{B} \times \left(\underline{\omega}_{IB}^{B} \times \underline{r}_{2,1}^{B}\right) - 2\, \underline{\omega}_{IB}^{B} \times \left(\dot{\underline{r}}_{2,1}^{B}\right) \tag{12}$$

Equation (12) is a general expression describing the relative acceleration between two points $\left(\ddot{\underline{r}}_{2,1}^{B}\right)$, as a function of the relative acceleration between the points measured in the I frame $\left(\ddot{\underline{r}}_{2,1}^{I}\right)$, the relative A frame position and velocity $\underline{r}_{2,1}^{B}, \left(\dot{\underline{r}}_{2,1}^{B}\right)$, the angular orientation C_I^B of frame B relative to frame I, and $\underline{\omega}_{IB}^{B}$, the B frame measured angular velocity of frame B relative to the I frame.

We now invoke the inertial characteristic to frame I for which equation (3) applies. Substituting (3) into (12) finally obtains the general expression for the acceleration between two mass points as viewed in a coordinate frame rotating relative to an inertial I frame as a function of force and natural motion accelerations measured in the rotating frame:

$$\left(\ddot{\underline{r}}_{2,1}^{B}\right) = \underline{a}_{F_2}^{B} - \underline{a}_{F_1}^{B} + \Delta\underline{a}_{N_{2,1}}^{B}$$

$$- \dot{\underline{\omega}}_{IB}^{B} \times \underline{r}_{2,1}^{B} - \underline{\omega}_{IB}^{B} \times \left(\underline{\omega}_{IB}^{B} \times \underline{r}_{2,1}^{B}\right) - 2\, \underline{\omega}_{IB}^{B} \times \left(\dot{\underline{r}}_{2,1}^{B}\right) \tag{13}$$

Velocity and position in the rotating B frame are obtained from (1) and (2)

$$\left(\dot{\underline{r}}_{2,1}^{B}\right) = \int_0^t \left(\ddot{\underline{r}}_{2,1}^{B}\right) dt + \left(\dot{\underline{r}}_{2,1}^{B}\right)_0 \qquad \underline{r}_{2,1}^{B} = \int_0^t \left(\dot{\underline{r}}_{2,1}^{B}\right) dt + \underline{r}_{2,1_0}^{B} \tag{14}$$

Equation (13) with (14) relates the relative position and velocity between two mass points as computed in a rotating coordinate frame, to the force acceleration on each mass point, the difference in natural motion acceleration at the mass points, and the angular rotation rate of the computational coordinate frame relative to inertial coordinates.

Appendices C and D expand the results in this section to encompass the rotational dynamics of mass groups and rigid bodies. The results are identical to what has been obtained in the past using the traditional Newtonian formulation, e.g., [3 - Chapts. 5 & 6].

THE CONCEPT OF INERTIAL ROTATION

For both the Newtonian and the new revised formulation of motion, the definition of an inertial coordinate frame is one in which the laws of motion apply. For the new formulation, the laws of motion are as expressed by (3) and (5) with (4). Based on this definition, we can also define a measurable concept of inertial rotation:

> Inertial rotation is the angular rotation rate of a coordinate frame relative to an inertial coordinate frame.

We should also recognize that there can be more than one inertial coordinate frame. For example, consider a coordinate frame A that has zero angular rate relative to inertial frame I (i.e., C_I^A is constant). Under the constant C_I^A condition, multiplying equation (3) by C_I^A yields:

$$\left(\ddot{\underline{r}}_{2,1}^{A}\right) = \underline{a}_{F_2}^{A} - \underline{a}_{F_1}^{A} + \Delta\underline{a}_{N_{2,1}}^{A} \qquad (15)$$

Equation (15) is the revised law of motion, but in another inertial frame A.

Frame B in the previous section is a general example of a non-inertial coordinate frame because it has angular rate $\underline{\omega}_{IB}$ relative to inertial frame I. On the other hand, if $\underline{\omega}_{IB}$ is zero, the (13) motion equation reduces to

$$\left(\ddot{\underline{r}}_{2,1}^{B}\right) = \underline{a}_{F_2}^{B} - \underline{a}_{F_1}^{B} + \Delta\underline{a}_{N_{2,1}}^{B} \qquad (16)$$

which is revised law of motion equation (3) in frame B. Thus, by virtue of (16), frame B would then represent an inertial coordinate frame having (by definition) zero angular rate relative to any other inertial frame (e.g., I or A). Means for determining whether a particular selected frame is inertial or not, will be discussed subsequently based on measurable performance parameters.

INERTIAL SENSORS

Inertial sensors are instruments that can be used to measure the force acceleration and inertial angular rotation rate elements in equations (3) and (13). Accelerometers located at mass points 1 and 2 measure the force acceleration components at these locations. Gyros having input axes aligned with equation (13) frame A axes, measure the inertial angular rate components in (13).

Accelerometers

An accelerometer is a device that can be used to directly measure force-generated acceleration. Accelerometers implement (5) using a proof mass located in a body whose force acceleration is to be measured. The proof-mass position location is controlled by forces generated within the accelerometer to maintain a fixed location of the proof-mass within the body-mounted accelerometer case. The resulting proof-mass force acceleration is thereby controlled to equal the body's force acceleration. By dividing the measured accelerometer control force by the mass of the proof mass as in (5), a direct measurement of body force acceleration is obtained. Most accelerometers are designed to measure force acceleration along a single axis (the accelerometer input axis). Three accelerometers are then required to measure each of three components of the force acceleration vector.

Mechanical Gyros

A classical momentum wheel gyro consists of a spinning rotor "wheel" enclosed in a case. The rotor is mounted with a suspension mechanism that allows only known torques (differential equal magnitude forces operating at opposite ends of a lever arm) to be applied to the spinning mass perpendicular to axes for which case angular rate is to be measured. The governing equations of motion are (D-9) and (D-10) with coordinate frame A parallel to rotor fixed axes and frame B parallel to gyro case fixed axes:

$$\underline{\omega}_{IB} = \underline{\omega}_{I,Case} \qquad \underline{\omega}_{IA} = \underline{\omega}_{I,Case} + \underline{\omega}_{Case,Rtr} \tag{17}$$

$$\underline{H}^B_{cm} = J^B_{cm}\left(\underline{\omega}^B_{I,Case} + \underline{\omega}_{Case,Rtr}\right) \qquad J^B_{cm} = -\left[\int \left(\underline{r}^B_{i,cm}\times\right)\left(\underline{r}^B_{i,cm}\times\right)\rho_i \, dv\right] \tag{18}$$

$$\begin{aligned}
\left(\dot{\underline{H}}^B_{cm}\right) &= \left[J^B_{cm}\left(\underline{\omega}^B_{I,Case} + \underline{\omega}^B_{Case,Rtr}\right)\right]^{\cdot} \\
&= \left(\dot{J}^B_{cm}\right)\left(\underline{\omega}^B_{I,Case} + \underline{\omega}^B_{Case,Rtr}\right) + J^B_{cm}\left[\left(\dot{\underline{\omega}}^B_{I,Case}\right) + \left(\dot{\underline{\omega}}^B_{Case,Rtr}\right)\right] \\
&= \Sigma\left(\underline{r}^B_{i,cm} \times \underline{F}^B_{External_i}\right) - \underline{\omega}^B_{I,Case} \times \underline{H}^B_{cm} + \int \left(\underline{r}^B_{i,cm} \times \Delta\underline{a}^B_{N_{i,cm}}\right)\rho_i \, dv
\end{aligned} \tag{19}$$

where

$\underline{\omega}_{I,Case}$ = Angular rate of the gyro case relative to the inertial I frame.

$\underline{\omega}_{Case,Rtr}$ = Angular rate of the rotor relative to the I frame.

$\underline{r}_{i,cm}$ = Relative position vector from the rotor center-of-mass to point i on the rotor.

$\underline{H}_{cm}$ = Angular momentum of the rotor about its center of mass relative to the I frame coordinates.

J_{cm} = Moment of inertia tensor of the rotor.

dv = Differential rotor volume element at point i.

ρ_i = Mass density at rotor point i.

and the rotor center of mass is defined implicitly from (A-4) as the point on the rotor for which

$$\int \underline{r}_{i,cm}\, \rho_i \, dv = 0 \tag{20}$$

For a gyro used to measure the angular rate of a structure to which it is mounted, the gyro case rotates with the structure, and the torque $\sum\left(\underline{r}_{i,cm}^{B} \times \underline{F}_{External_i}^{B}\right)$ in (19) would be generated within the gyro to maintain spin axis alignment with gyro case fixed axes. Then from symmetry, the rotor mass distribution will not change in the case aligned B frame, and the rate of change of the moment of inertial tensor B frame projection $\left(\dot{J}_{cm}^{B}\right)$ in (19) will be zero. Additionally, the gyro rotor spin rate relative to the case would be controlled by a synchronous hysterisis drive motor to remain constant, hence $\left(\dot{\underline{\omega}}_{Case,Rtr}^{B}\right)$ will also be zero. Thus, with rearrangement, (19) simplifies to

$$\begin{aligned}\underline{T}^{B} &= \underline{\omega}_{I,Case}^{B} \times \underline{H}_{Rtr}^{B} \\ &+ J_{cm}^{B}\left(\dot{\underline{\omega}}_{I,Case}^{B}\right) + \underline{\omega}_{I,Case}^{B} \times \left(J_{cm}^{B}\, \underline{\omega}_{I,Case}^{B}\right) - \int \left(\underline{r}_{i,cm}^{B} \times \Delta\underline{a}_{N_{i,cm}}^{B}\right) \rho_i \, dv\end{aligned} \tag{21}$$

with the torque $\underline{T}$ applied on the rotor and the rotor angular momentum $\underline{H}_{Rtr}$ relative to the case defined in B frame coordinates as

$$\underline{T}^{B} \equiv \sum\left(\underline{r}_{i,cm}^{B} \times \underline{F}_{External_i}^{B}\right) \qquad \underline{H}_{Rtr}^{B} \equiv J_{cm}^{B}\, \underline{\omega}_{Case,Rtr}^{B} \tag{22}$$

The last three terms in (21) are generally small compared to the others, in part due to the largeness of the angular momentum $\underline{H}_{Rtr}$ designed into the gyro rotor (by spin rate and mass). Thus, (21) shows that the torque applied to the rotor is approximately proportional to the inertial angular rate of the gyro case $\underline{\omega}_{I,Case}^{B}$. Using appropriate scaling, an electrical signal proportional to the applied torque can thereby be formed as the gyro output measurement of $\underline{\omega}_{I,Case}^{B}$. For improved accuracy in measuring $\underline{\omega}_{I,Case}^{B}$, the gyro output can be corrected for the $J_{cm}^{B}\left(\dot{\underline{\omega}}_{I,Case}^{B}\right)$ and $\underline{\omega}_{I,Case}^{B} \times \left(J_{cm}^{B}\, \underline{\omega}_{I,Case}^{B}\right)$ terms in (21) by appropriate software compensation routines applied in computers using the gyro output (i.e., as a function of the rescaled $\underline{T}^{B}$ gyro input to the computer). The $\int \left(\underline{r}_{i,cm}^{B} \times \Delta\underline{a}_{N_{i,cm}}^{B}\right) \rho_i \, dv$ gravity gradient term in (21) is generally negligible compared with $\underline{T}^{B}$. It is to be noted, however, that if the rotor is designed to have a symmetrical mass distribution (i.e., diagonal inertia tensor J_{cm}^{B} with equal diagonal element components - see (18) for J_{cm}^{B} definition), it can be shown that the gravity gradient term would be zero [3 - Sect. 6-4] (as would the $\underline{\omega}_{I,Case}^{B} \times \left(J_{cm}^{B}\, \underline{\omega}_{I,Case}^{B}\right)$ term).

Optical Gyros

Optical gyros measure angular rotation based on the inertial properties of light traveling in a closed path (i.e., the Sagnac effect). To help understand the Sagnac principle, consider a rotating rigid body within which is contained an empty closed circular tube ring of radius r, with the body rotation axis perpendicular to the plane formed by the closed circular tube. For simplicity, consider the case where forces applied to the body are perpendicular to the plane of the tube ring. Imagine a small frictionless Newtonian proof mass contained within the tube that is given an initial velocity v relative to the tube at a reference point marker on the tube. To an observer stationed on the rotating body, the proof mass would appear to traverse the tube path at constant velocity v before returning to the reference point. This result would be the same whether or not the body was experiencing a rotational rate during the proof mass transit. However, if the body is rotating at angular rate ω during the transit along the tube, the proof-mass velocity measured by a non-rotating observer would equal $v \pm \omega\, r$ (plus if v is in the direction of rotation and minus otherwise).

Now consider that the proof-mass is a photon of light traveling at velocity c relative to the reference point on the circular tube. Curiously, the velocity of the proof-mass relative to "non-rotating" inertial coordinates would then also be c, i.e., not $c + \omega\, r$ as what might have been inferred by one unfamiliar with relativity theory. In other words, to a non-rotating observer, the velocity of the photon around the tube would be the same as it would have been if the body was not rotating, even if the test photon was generated within the rotating body during the rotation. This interesting result is the well-known Sagnac effect and is the basis for measuring angular rotation rate relative to a non-rotating inertial reference frame using optical rotation sensors.

What is further curious is that Sagnac based optical rate sensors measure angular rate relative to the same "non-rotating" inertial frame of reference as mechanical gyros. This article has discussed a linkage between Newtonian mechanics and a definable "non-rotating" inertial frame of reference. However, it is unclear to me why the Sagnac optical gyro "non-rotating" frame of reference is the same as that for mechanical gyros. This is certainly the case as has been borne out by experiment and use over many years. To not belabor this point, for the remainder of this discussion, "rotation" will mean the same inertial angular rotation measured by mechanical gyros.

Using The Sagnac Effect To Sense Rotation - Consider that the optical path in the previous discussion is circular with radius r, and the optical path length is s for one complete circuit from the reference point. The time interval for a photon of light to traverse the optical path would be $\Delta t = s / c$. During this time interval, body rotation would cause the reference point to traverse a distance $\Delta s = \omega\, r\, \Delta t = \omega\, r\, s / c$. The effect in the direction of rotation is that when the photon completes a full path traversal, it would have to travel an additional Δs distance to reach the reference point. For the photon being a part of a single frequency light beam, this means that when the light beam returns to the reference point, it will be shifted in phase from what it would have been without the rotation. The amount of phase shift would be $\Delta s \ / \ \lambda = \omega \ r \ s \ / \ (c \ \lambda)$ where λ is the light beam wave length. For a light beam traveling with the rotation, the phase shift would be negative. For a light beam traveling against the rotation, the phase shift would be positive. The difference in phase shifts between oppositely directed light beams would be $2 \ \omega \ r \ s \ / \ (c \ \lambda)$. The phase shift difference is what is detected for output by optical rotation sensors.

There are two types of optical rotation sensors, each based on the Sagnac effect; fiber optic gyros and ring laser gyros.

<u>Fiber Optic Gyros</u> - In a fiber optic gyro (FOG), a near single frequency beam of light is generated with a Galium Arsenide diode within the FOG and transmitted into an optical fiber through a fiber-optic coupler. Fiber optic splicing is then used to split the fiber (and light beam) into two segments, each attached to opposite ends of a fiber optic coil enclosed within the FOG (Several hundred meters of fiber typically wound into a 2 - 3 inch diameter coil). A closed optical path is thereby created by the fiber coil containing two oppositely directed beams of light, one "clockwise", the other "counter-clockwise". After completing a circuit around the fiber optic path, the beams return to the splice point where they optically recombine, then leave the coil through a second splice onto a photodiode optical detector. If the return beams are in phase, the power in the two beams add, providing a maximum output voltage from the detector. Under angular rotation around the fiber coil axis, the Sagnac effect will generate a phase difference between the clockwise and counter-clockwise return beams which alters the combined beam power level on the photodiode. The altered power level changes the magnitude of the phototdiode output which then becomes the detected measure of angular rate for the FOG.

<u>Ring Laser Gyros</u> - In a ring laser gyro (RLG), the closed optical path is a sealed continuous cavity filled with a mixture of helium/neon gas at low pressure. Ionizing the gas generates two laser beams, one clockwise, the other counter-clockwise, each reflected around the beam path by dielectric mirrors (3 or 4, depending on the RLG design configuration). As with the FOG, for each circuit of the beams around the closed path, the Sagnac effect generates a phase difference between the counter-rotating beams proportional to the RLG angular rotation rate. Unlike the FOG, however, for each passage of the beams through the ionized helium/neon gas, new photons at beam frequency are generated that are in phase with each beam, thereby increasing the power in each beam ("gain"). The added photons compensate for beam energy loss in the cavity (kept to a minimum by careful design/mnufacturing practice). Most importantly, the gain allows the beams to continue making closed circuits around the optical path indefinitely. As a result, for each circuit around the beam path, an additonal phase shift is created, generating a phase difference between the beams that grows linearly with time. Thus, the phase difference in the RLG becomes proportional to the integral of angular rate (in contrast with the FOG where it is proportional to angular rate).

The RLG output is generated by allowing a small portion of each beam to escape through one of the mirrors, then merged together by a combiner reflecting prism onto a photodiode detector. The combined beams creates a sinusoidal electrical output from the photodiode, each output wave passage representing a 180 degree change in the optical phase difference between the laser beams. Counting the output waves generates a digital signal proportional to the integrated angular rate. The direction of rotation can be ascertained by designing the countering beam combiner prism so that the beams merge at a small angle relative to one another (wedge angle). This creates an optical fringe pattern in space that moves across the beams at a rate proportional to the RLG input rate. Passage of each fringe across the beams corresponds to a photodiode sinusoidal output cycle. By mounting two photodiodes in the combined beam optical fringe space, one physically separated from the other by one quarter of a fringe, the sinusoidal output from one thereby becomes 90 degrees phase shifted from the other. Which output is leading or lagging the other determines the direction of rotation.

It is also informative to imagine viewing the RLG counter-rotating laser beams from the perspective of an observer stationed on a "non-rotating" platform. To such an observer, the clockwise and counter-clockwise traveling wave laser beams would merge into a single standing wave beam of light, with the distance between nodes equal to half the individual laser beam wavelength (0.62 micron wavelength for modern-day RLGs). The RLG optical fringe output pattern can then be interpreted as the equivalent to viewing the standing wave by an observer stationed on the rotating gyro, thereby observing standing wave passage through a viewing window, each wave passage corresponding to the passage of one optical fringe across the viewing window (measured by the photodiodes). This is another statement of the Sagnac effect relative to a non-rotating observer: maintaining a closed stationary standing wave beam of light in the presence of rotation of the medium in which the light beam is created (for a FOG) or sustained (for an RLG).

SELECTION OF A PARTICULAR INERTIAL COORDINATE FRAME

Since the inertial coordinate frame axes in the revised formulation are only used for angular referencing, inertial coordinates can be defined based on observable remote celestial formations. The only requirement is that the projections of relative acceleration between masses on the selected inertial frame axes represent equation (3) or its equivalent (A-6), (B-4), (C-7), (D-6) or (13) (with $\underline{\omega}_{IB}^{I} = 0$). Either of these equations shows that the relative velocity rate in the selected I frame must account for only relative force and natural motion acceleration. Since the reformulated definition of inertial coordinates only requires the angular orientation of its unit vectors, the previous statement can be simplified to state that the angular orientation of the I frame unit vectors must be such as to satisfy the previous equations. Moreover, each inertial unit vector definition can be made independently from the others, so long as the orthogonality constraint between the three is satisfied. (This is easily accomplished for example, by first defining a unit vector pair based on celestial observations. One of the I frame unit vectors can than be constructed as the normalized perpendicular to the observed pair. A second I frame unit vector can then be defined as one of the observed pair. The third I frame unit vector would then be formed as the cross-product between the first selected two.)

Observations of celestial relative mass motion generally entail observations of relative position between masses (e.g., the double integral of relative acceleration equation (3) as in (4)). For example, consider an inertial coordinate frame with one of its unit vectors parallel to a line between two observable and easily identifiable stars. Assuming that the selected stars are reasonably close to one another it can be assumed that the natural motion acceleration created from other celestial elements at each star location will be the same. The stars can also be selected so that the relative distance is sufficiently far that the gravitational acceleration generated by one on the other is negligibly small. Finally, it is safe to approximate that the force acceleration acting on each star will be zero. Thus, (3) shows that the projection of the relative star acceleration motion along inertial axes will be zero. Hence, if the initial relative velocity between the stars is negligibly small, (4) shows that the line between the stars will be constant and suitable for selection as a reference direction for one of the inertial frame unit vectors. (Note: If the relative velocity between the stars has a component perpendicular to the line between the stars, the line will rotate in inertial space, hence, would not be suitable for use as an inertial coordinate axis direction vector. Celestial observations of relative star locations from earth has shown that after correction for earth's rotation, the distance vector between stars remains constant in inertial coordinates).

As another example based on observed relative mass movement, consider the motion of an orbiting planet around a star. Call the star mass 1 and the planet mass 2. Then from (3) and (4):

$$\frac{d^2}{dt}\underline{r}^I_{2,1} = \underline{a}^I_{F_2} - \underline{a}^I_{F_1} + \Delta\underline{a}^I_{N_{2,1}} \qquad \frac{d}{dt}\underline{r}^I_{2,1} = \int_0^t \left(\frac{d^2}{dt}\underline{r}^I_{2,1}\right)dt + \left(\frac{d}{dt}\underline{r}^I_{2,1}\right)_0 \qquad (23)$$

In celestial space, the $\underline{a}_{F_1}$, $\underline{a}_{F_2}$ force acceleration can be safely approximated as zero. Assuming a large distance of the planet from the star, each with general mass symmetry, (6) would show that the star's generated gravitational component of natural acceleration on the planet would be $\underline{a}_{N/g_2} = -\, m_1 \frac{\mu}{r_{21}^3}\underline{r}_{21}$. Similarly, the planet's generated gravitational component of natural acceleration on the star would be $\underline{a}_{N/g_1} = -\, m_2 \frac{\mu}{r_{21}^3}\underline{r}_{21}$. It can also be assumed that the selected star and planet will be sufficiently distant from other natural motion generating mechanisms that their values at the star and planet locations are equal, hence cancel in (23). Thus, (23) simplifies to

$$\frac{d^2}{dt}\underline{r}^I_{2,1} = -\left(m_2 - m_1\right)\frac{\mu}{r_{21}^3}\underline{r}^I_{2,1} \qquad \frac{d}{dt}\underline{r}^I_{2,1} = \int_0^t \left(\frac{d^2}{dt}\underline{r}^I_{2,1}\right)dt + \left(\frac{d}{dt}\underline{r}^I_{2,1}\right)_0 \qquad (24)$$

Define vector $\underline{w}^I$ normal to $\underline{r}^I_{2,1}$ and $\frac{d}{dt}\underline{r}^I_{2,1}$ as a potential direction for an I frame unit vector:

$$\underline{w}^I \equiv \underline{r}^I_{2,1} \times \frac{d}{dt}\underline{r}^I_{2,1} \tag{25}$$

(Note that $\underline{w}^I$ is a mass normalized relative angular momentum of a point mass at 2 relative to point 1 as in (C-1) of Appendix C - i.e., per unit point 2 mass.) The rate of change of $\underline{w}^I$ in the I frame is

$$\frac{d}{dt}\underline{w}^I = \left(\frac{d}{dt}\underline{r}^I_{2,1}\right) \times \left(\frac{d}{dt}\underline{r}^I_{2,1}\right) + \underline{r}^I_{2,1} \times \frac{d^2}{dt}\underline{r}^I_{2,1} = -\,\underline{r}^I_{2,1} \times \left[(m_2 + m_1)\frac{\mu}{r^3_{21}}\underline{r}^I_{2,1}\right] = 0 \tag{26}$$

Thus $\underline{w}^I$ in (25) will remain constant at its initial value in the I frame, and a unit vector parallel to (26) will have the required characteristic for an I frame unit vector. Note also that the component of $\frac{d^2}{dt}\underline{r}^I_{2,1}$ in (24) along $\underline{w}^I$ is proportional to

$$\left(\frac{d^2}{dt}\underline{r}^I_{2,1}\right) \cdot \underline{w}^I = -(m_2 + m_1)\frac{\mu}{r^3_{21}}\underline{r}^I_{2,1} \cdot \left(\underline{r}^I_{2,1} \times \frac{d}{dt}\underline{r}^I_{2,1}\right) = 0 \tag{27}$$

Hence, the relative velocity $\frac{d}{dt}\underline{r}^I_{2,1}$ will remain in a fixed plane defined by the constant $\underline{w}^I$ normal to the plane of motion (i.e., the orbit plane of the planet around the star). As such, a unit vector perpendicular to the observed planet orbital plane (i.e., parallel to $\underline{w}^I$ in (25)) can be the basis for selection of one of the I frame unit vectors. A perpendicular to earth's orbital plane around the sun is an example of this approach applied in the past.

Based on the revised formulation of motion, Appendix D shows that in the absence of applied torques, the angular momentum of a rigid body about its center-of-mass will remain constant in an inertial coordinate frame. As such, a unit vector directed along a torque free angular momentum vector can be used as an axis of an inertial frame. As an example of a rigid body angular momentum based approach, consider the angular momentum $\underline{H}_{cm}$ of a rotating planet about its center of mass based on (D-6):

$$\frac{d\underline{H}^{\mathbf{I}}_{cm}}{dt} = \sum\left(\underline{r}^I_{i,cm} \times \underline{F}^I_{External_i}\right) + \int \left(\underline{r}^I_{i,cm} \times \Delta\underline{a}^I_{N_{i,cm}}\right) \rho_i \, dv \tag{28}$$

Forces applied to the planet can be approximated to be zero. As in the previous example, consider that the planet is orbiting a star. If the planet is sufficiently removed from the star center, the gravitational acceleration gradient $\Delta\underline{a}^I_{N_{i,cm}}$ across the planet produced by the star will be very small. For a symmetrical planet mass distribution, it can be shown that the composite effect in (28) of $\Delta \underline{a}^{I}_{N_{i,cm}}$ on $\dfrac{d\underline{H}^{\mathbf{I}}_{cm}}{dt}$ will be zero. If the planet has only a small amount of mass asymmetry, the natural motion acceleration generated by $\Delta \underline{a}^{I}_{N_{i,cm}}$ will then be negligible. Finally, consider that the natural motion acceleration gradient across the planet generated from other celestial elements is negligible. Thus, (28) show that under these conditions, the rate of change of the angular momentum of the planet around its center of mass $\underline{H}_{cm}$ will be zero when measured in inertial coordinates. Consequently, $\underline{H}^{\mathbf{I}}_{cm}$ will be constant, hence suitable for use as a reference direction for one of the unit vectors in an inertial coordinate frame. A unit vector along earth's rotation axis is an example of this approach applied in the past.

Unit vectors for an inertial coordinate frame can also be constructed artificially using gyros. Three gyros mounted to a common base measure the angular rate vector of the mounting base relative to inertial coordinates. Equation (9) can be integrated in a computer to calculate the angular orientation of the gyro mount relative to inertial I frame coordinates (in the form of direction cosine matrix C^{B}_{I}). The columns of C^{B}_{I} are the I frame unit vector components as measured in a coordinate frame B aligned with the gyro input axes. Such an implementation is the basis for inertial navigation systems (Appendix E) that relate force acceleration measurements (from accelerometers mounted with the gyros on the same base) into their equivalent I frame components for integration into relative velocity and position - based on (3) and (4).

APPLYING THE NEW FORMULATION TO DESCRIBE COMMON PHENOMENA

Examples follow showing how the new relative motion formulation explains common phenomena, yielding identical results as classical formulations of the past.

The Concept Of Weight

Consider a coordinate frame B that rotates with the earth. Consider a point 1 to be at earth's center of mass. Because the earth is in free space, the net force $\underline{F}_1$ impinging on it is zero, hence, the point 1 force acceleration will be zero. Assuming an approximate symmetric mass distribution for the earth, application of (6) would show that the gravitational acceleration at earth's center point 1 due to earth's mass is zero (Note: the same can be deduced from symmetry, recognizing that the gravitational field generated at earth's center from an arbitrary mass point in the earth is exactly equal in magnitude and oppositely directed from the equivalent mass on the opposite side of the center of mass). Now consider a fixed point 2 on earth's surface. Call $\underline{a}_{N_{2/g-Earth}}$ the gravitational acceleration caused by earth mass at point 2. In addition to earth mass effects, points 1 and 2 have natural motion acceleration produced by the universe. Call the difference between these at points $\Delta \underline{a}_{N_{2,1-\text{Universe}}}$. (Note - $\Delta \underline{a}^B_{N_{2,1-\text{Universe}}}$ is generated primarily by near earth spatial masses, the moon and secondarily, the sun). Substituting these factors in (13) yields in the B frame

$$\begin{aligned} \left(\ddot{\underline{r}}^B_{2,1}\right) &= \underline{a}^B_{F_2} + \underline{a}^B_{N_{2/g-\text{Earth}}} + \Delta \underline{a}^B_{N_{2,1-\text{Universe}}} \\ &\quad - \dot{\underline{\omega}}^B_{IB} \times \underline{r}^A_{2,1} - \underline{\omega}^B_{IB} \times \left(\underline{\omega}^B_{IB} \times \underline{r}^B_{2,1}\right) - 2\, \underline{\omega}^B_{IB} \times \left(\dot{\underline{r}}^B_{2,1}\right) \end{aligned} \tag{29}$$

Because point 2 is fixed on earth's surface, the relative acceleration $\left(\ddot{\underline{r}}^B_{2,1}\right)$ and velocity $\left(\dot{\underline{r}}^B_{2,1}\right)$ between points 1 and 2 is zero. Additionally, earth's angular rate relative to inertial coordinates is constant, hence, for the B frame that rotates with the earth, $\dot{\underline{\omega}}^B_{IB} = 0$. Substitution in (29) and solving for $\underline{a}^B_{F_2}$ then yields

$$\underline{a}^B_{F_2} = -\, \underline{a}^B_{N_{2/g-\text{Earth}}} + \underline{\omega}^B_{IB} \times \left(\underline{\omega}^B_{IB} \times \underline{r}^B_{2,1}\right) - \Delta \underline{a}^B_{N_{2,1-\text{Universe}}} \tag{30}$$

A mass body located at point 2 would experience the $\underline{a}^B_{F_2}$ force acceleration of (30) which is directed approximately upward, opposite from earth's downward gravity $\underline{a}^B_{N_{2/g-\text{Earth}}}$. The magnitude approximately equals the magnitude of $\underline{a}^B_{N_{2/g-\text{Earth}}}$ because the centripetal acceleration term $\underline{\omega}^B_{IB} \times \left(\underline{\omega}^B_{IB} \times \underline{r}^B_{2,1}\right)$ is much smaller than $\underline{a}^B_{N_{2/g-\text{Earth}}}$, and because $\Delta \underline{a}^B_{N_{2,1-\text{Universe}}}$ is much smaller than $\underline{\omega}^B_{IB} \times \left(\underline{\omega}^B_{IB} \times \underline{r}^B_{2,1}\right)$. From (5), the $\underline{a}^B_{F_2}$ inertial acceleration would be created by a force $\underline{F}_2$ operating on a stationary mass body located at point 2. This is the quantity denoted as "weight". When we "weigh" a body, we are measuring the upward force $\underline{F}_2$ using an appropriately calibrated scale.

<u>Measuring Weight On Earth's Surface, Other Planets, And In Space</u>

Two types of scales can be used to measure weight, a force-balance type and a mass-balance type. For the former, a spring-type mechanism is used to provide the balance force $\underline{F}_2$. The scale readout is proportional to the spring deflection under $\underline{F}_2$ which is calibrated to provide the proper unit of force per unit mass in earth's gravity field. In the English system, one pound is the force required to inertially accelerate one slug of mass at one foot per second-squared. On a planet other than earth, a force-balance scale would have a scale factor error equal to the ratio of gravity magnitude values on the planet compared to that of the earth (e.g., the reported weight of an earth calibrated scale on Mars would be 0.45 of its value on the earth, i.e., the ratio between Mars/Earth surface gravity).

For a mass balance type scale, a test mass is used with a suitable lever arm mechanism to balance the mass being weighed. Both the test mass and the mass being weighed generate an inertial acceleration, hence reaction force proportional to the local gravity magnitude. Balance is achieved by adjusting the test mass lever-arm position on the balancing mechanism so that the lever-arm/force products for the two masses become equal. The resulting lever arm position at the balance point thereby becomes proportional to the ratio of the balance forces (i.e., proportional to the ratio of the masses) because both are in the same gravity field. The lever arm position is calibrated based on the test mass size to provide an output in acceptable "weight" units (e.g., one pound per slug per unit of earth's gravity). Notice, however, that because of the mass balance principal underlying this type of scale, the scale would have the same output on any planet as it would on earth's surface. Thus, the balance type scale is actually a mass measurement device whose output is calibrated in pounds rather then slugs to correspond with the reaction force it would produce on earth's surface. The concept of "pound mass" rather than "inertial mass" has been used to "clarify" the distinction between mass being weighed and mass being inertially accelerated. However, according to this revised formulation, there is no need to distinguish one from the other; both are based on the force applied to inertial mass in generating inertial acceleration.

Now consider a body being weighed in free space. Consider the body being weighed to be at location point 2 and the scale to be at point 1 in a non-rotating spacecraft under unforced natural motion. Because the body and scale will be stationary relative to one another, the relative acceleration between them $\frac{d^2}{dt}\underline{r}^I_{2,1}$ is zero. Because the distance between them is small, their difference in natural motion acceleration $\Delta\underline{a}^I_{N_{2,1}}$ will be essentially zero. From (3), the difference in the force accelerations ($\underline{a}^I_{F_2} - \underline{a}^I_{F_1}$) will then be zero. Since it has been stipulated that the bodies and the spacecraft containing the bodies are in free space, there will be no external force applied to either, thus, $\underline{a}^I_{F_2}$ and $\underline{a}^I_{F_1}$ will also be zero. For a force-balance type scale, there will thereby be zero force applied by the scale to the body being weighed, and the measured force (and deduced weight) will be zero (i.e., "weightless"). Curiously, however, for a mass-balance type scale, the output will be indeterminate because any lever arm position will be a balance point. If the spacecraft containing the body and scale has a small amount of force applied to it (e.g., from a reaction jet), both $\underline{a}^I_{F_2}$ and $\underline{a}^I_{F_1}$ will be generated, and a single balance point, thereby created, corresponding to a correct earth referenced "pound-mass" output.

Einstein's Elevator Thought Experiment

Einstein's elevator thought experiment [2 - pp. 75 - 79] deals with the difference between observations of two observers, one stationary on the earth, the other in an enclosed capsule in Newtonian gravity free inertial space, being pulled upward by a rope at an acceleration equal to earth's surface gravity magnitude. The observer in the capsule sees free objects within the capsule appear to be accelerating downward at the acceleration of gravity, exactly as would an earthbound observer see objects that are dropped to fall freely to the earth under earth's gravitational pull. Einstein concluded that the two situations are equivalent and that the observer in the elevator would interpret his observation as being created by a "uniform" gravity field, much as would the earthbound observer.

What does revised formulation (3) predict for the result of this experiment? First consider the earth fixed observer at point 2 on the earth's surface where $\underline{a}_{F_2}^{B}$, force acceleration (in a B frame rotating with the earth), is given by (30). For simplicity (as in Einstein's model) we will ignore the $\underline{\omega}_{IB}^{B}$ earth rate effect in (30) so that the B frame can be considered to be inertial frame I, hence, (30) becomes

$$\underline{a}_{F_2}^{I} = -\left(\underline{a}_{N_{2/g-\mathrm{Earth}}}^{I} + \Delta\underline{a}_{N_{2,\mathrm{ErthCntr-Universe}}}^{I}\right) \tag{31}$$

where

$\Delta\underline{a}_{N_{2,\mathrm{ErthCntr-Universe}}}^{I}$ = The difference between the natural acceleration of the universe at surface point 2 compared to a point at earth's center.

Consider point 2 to be close to another point 1 in free-fall at the point 2 location. Then $\Delta\underline{a}_{N_{2,1}}^{I}$ and the $\underline{a}_{F_1}^{I}$ force acceleration is zero, and (31) in (3) shows that $\frac{d^2}{dt}\underline{r}_{2,1}^{I}$, the acceleration of point 2 relative to point 1 would be

$$\frac{d^2}{dt}\underline{r}_{2,1}^{I} = -\left(\underline{a}_{N_{2/g-\mathrm{Earth}}}^{I} + \Delta\underline{a}_{N_{2,\mathrm{ErthCntr-Universe}}}^{I}\right) \tag{32}$$

By definition, the acceleration $\frac{d^2}{dt}\underline{r}_{1,2}^{I}$ of free-fall point 1 relative to point 2 is the negative of (32), hence,

$$\frac{d^2}{dt}\underline{r}_{1,2}^{I} = \left(\underline{a}_{N_{2/g-\mathrm{Earth}}}^{I} + \Delta\underline{a}_{N_{2,\mathrm{ErthCntr-Universe}}}^{I}\right) \tag{33}$$

Thus, free-fall point 1 falls relative to earth fixed point 2 at the combined natural acceleration of earth fixed point 2.

For an observer at point 3 within the rope accelerated capsule, the experiment sets the applied rope force acceleration $\underline{a}^{I}_{F_3}$ equal to stationary observer 2 value $\underline{a}^{I}_{F_2}$, or from (31),

$$\underline{a}^{I}_{F_3} = -\left(\underline{a}^{I}_{N_{2/g-\mathrm{Earth}}} + \Delta\underline{a}^{I}_{N_{2,\mathrm{ErthCntr-Universe}}}\right) \tag{34}$$

Now consider a mass point 4 to be in free-fall within the capsule. As in the previous discussion for the earth fixed observer, both $\Delta\underline{a}^{I}_{N_{3,4}}$ and the $\underline{a}^{I}_{F_4}$ force acceleration is zero. Substitution in (3) then yields the acceleration of free-fall point 4 relative to the accelerating capsule point 3:

$$\frac{d^2}{dt}\underline{r}^{I}_{4,3} = -\,\underline{a}^{I}_{F_3} = \left(\underline{a}^{I}_{N_{2/g-\mathrm{Earth}}} + \Delta\underline{a}^{I}_{N_{2,\mathrm{ErthCntr-Universe}}}\right) e \tag{35}$$

which is exactly the same as the (33) result for the earth fixed observer.

Einstein interpreted these results as demonstrating that natural and artificially generated gravitational acceleration are indistinguishable, provided that there is an equivalency between inertial and gravitational mass. His conclusion was based on the interpretation of gravitational acceleration as being the result of an applied gravitational force. Gravitational mass was then required to translate gravitational acceleration into the gravitational force that created it. But with the new formulation, the concept of gravitational mass never enters the analysis. Inertial mass is required, but only implicitly as represented in (5) for translating the applied force acceleration to the force that produces it. Both the earth and capsule bound observers measure the same effect; the same applied force acceleration that accelerates the observer from the free-fall mass being observed. For the earth-bound case the force acceleration is the natural reaction of the earth surface against gravitational free-fall. For the capsule-bound case, the force generated acceleration is created to match the same gravity resisting force acceleration experienced on earth's surface. Neither case measures gravitational acceleration, either actual or artificial as postulated by Einstein.

Free-Fall In Space And Earth Orbit

"Free-fall" has been defined as motion due to gravity alone (i.e., with no additional applied force) relative to a reference point commonly selected to be the center of some near-by planet or star (for which the applied force acceleration can be approximated as zero). From (3), the free-fall motion of a general point 2 relative to some non-forced acceleration reference point 1 is

$$\frac{d^2}{dt}\underline{r}^I_{2,1} = \Delta\underline{a}^I_{N_{2,1}} \tag{36}$$

The center of the earth has been a common reference point for referencing near-earth motion. From (3) for $\Delta\underline{a}^I_{N_{2,1}}$ in (36), the free-fall motion of a mass point 2 relative to point 1 at the center of the earth is:

$$\frac{d^2}{dt}\underline{r}^I_{2,1} = \underline{a}^I_{N_{2/g-\mathrm{Earth}}} + \Delta\underline{a}^I_{N_{2,1-\mathrm{Universe}}} \tag{37}$$

Integration of (4) for velocity and position with (37) as input would describe an orbit around the earth, depending on the initial velocity/position values (i.e., too high an initial velocity would generate an escape trajectory from the earth; too low or a misdirected initial velocity would produce a trajectory that impacted the earth). Equation (37) motion can be "artificially" generated by applying control forces to a vehicle that are designed to cancel all other applied forces, thereby controlling $\underline{a}^I_{F_2}$ to zero (e.g., by adjusting the thrust and lift in an airplane to cancel aerodynamic forces). The control sensor for such an application would be an accelerometer, measuring the inertial acceleration controlling the vehicle to maintain an average zero output during the free-fall maneuver. Note that in the previous example, local gravity is a primary element in point 2 motion, and the point 2 "environment" is not one of "zero or micro-gravity" as it is sometimes erroneously defined.

<u>Inertial Navigation</u>

An inertial navigation system (INS) is a portable autonomous device that calculates velocity and position between two points (1 and 2) by integrating relative acceleration between the points. The force portion $\underline{a}_F$ of the acceleration is measured by accelerometers. The natural component of acceleration $\underline{a}_N$ is gravity $\underline{a}_{N/g}$ based on an analytical model function of calculated position. The basic concept is represented in an inertial coordinate I frame by equations (3), (4), and (6). Gyros are used to determine the angular orientation of the accelerometers relative to inertial coordinates based on the integrated form [4 - pp. 3-53]:

$$\underline{a}^I_F = C^I_B\,\underline{a}^B_F \qquad\qquad C^I_B = \int \left(\underline{\omega}^B_{IB}\times\right)dt + \left(C^I_B\right)_0 \tag{38}$$

where the B frame is aligned with the accelerometer mount, $\underline{a}^I_F$ is INS force acceleration in inertial I frame coordinates, $\underline{a}^B_F$ is the accelerometer output vector, and $\underline{\omega}^B_{IB}$ is the B frame angular rate relative to the I frame as measured by gyros installed on the accelerometer mount. Means are also provided relating I frame coordinates to standard frames commonly used for navigation data representations (e.g., locally level coordinates aligned with north, east, vertical axes).

Typical versions of (3), (4), (6), and (38) implemented in a terrestrial INS calculate velocity relative to a rotating earth in a coordinate frame that remains locally level at the INS position as it moves relative to earth's surface. Position relative to the earth is determined from the velocity in terms of latitude/longitude angular units and altitude, using appropriate integration routines. The details are described in Appendix E.

CONCLUSIONS

The revised definition of natural motion and altering forces simplifies the understanding of commonly experienced natural phenomena by eliminating the need for gravitational force and associated gravitational mass. The basis for the new approach is dividing acceleration into two clearly defined parts, inertial acceleration generated by forces, and natural acceleration (including gravity) that governs unforced natural motion. Inertial acceleration is an absolute quantity measurable with accelerometers at any spatial location. Natural acceleration is a relativistic property of space that can only be measured (e.g., by optical means) as the relative difference between values at separated spatial locations. A corollary is that absolute natural motion acceleration at any spatial point is immeasurable. Application of the new approach provides a measurable definition of inertial coordinates, the angular orientation reference of all inertial angular-rate sensing instruments.

APPENDIX A

TRANSLATIONAL DYNAMICS OF MASS GROUPS

For a general mass point i within a group of masses, (3) with (5) relative to an arbitrary reference point 0 is

$$\frac{d^2}{dt}\underline{r}^I_{i,0} = \underline{a}^I_{F_i} - \underline{a}^I_{F_0} + \Delta\underline{a}^I_{N_{i,0}} \qquad \underline{a}^I_{F_i} = F_i \,/\, m_i \qquad \underline{a}^I_{F_0} = F_0 \,/\, m_0 \qquad \text{(A-1)}$$

Multiplying by m_i and summing over the group of i masses yields:

$$\sum m_i \frac{d^2}{dt}\underline{r}^I_{i,0} = \sum\left(\underline{F}_i + m_i\,\Delta\underline{a}^I_{N_{i,0}}\right) - M\,\underline{a}^I_{F_0} \qquad M = \sum m_i \qquad \text{(A-2)}$$

where M is the total inertial mass of the group. Newton's third law states that for every force there is an equal and opposite reaction force [1 - pp. 417]. Thus, the components of $\underline{F}_i$ due to point-to-point interaction within the group will sum to zero. Then (A-2) becomes

$$\sum m_i \frac{d^2}{dt} \underline{r}^I_{i,0} = \sum \underline{F}_{External_i} + \sum m_i \, \Delta\underline{a}^I_{N_{i,0}} - M \, \underline{a}^I_{F0} \qquad \text{(A-3)}$$

and $\underline{F}_{External_i}$ is the external force acting on the group at point i, exclusive of forces produced by mass interactions within the group. Equation (A-3) can be simplified by introducing the standard definition of "center-of mass" $\underline{r}_{cm/0}$ position relative to point 0 as

$$\underline{r}_{cm,0} = \frac{\sum m_i \, \underline{r}_{i,0}}{M} \qquad \text{(A-4)}$$

Successive differentiation of (A-4) in the I frame using general equations (10) for frame I position/velocity representation gives

$$\frac{d^2}{dt} \underline{r}^I_{cm,0} = \frac{1}{M} \sum \left(m_i \, \frac{d^2}{dt} \underline{r}^I_{i,0} \right) \qquad \text{(A-5)}$$

With (A-3) and rearrangement, (A-5) assumes the simplified form

$$\begin{aligned} M \frac{d^2}{dt} \underline{r}^I_{cm,0} &= \sum \underline{F}_{External_i} + \sum m_i \, \Delta\underline{a}^I_{N_{i,0}} - M \, \underline{a}^I_{F0} \\ &= \sum \underline{F}_{External_i} + \sum m_i \left(\Delta\underline{a}^I_{N_{i,cm}} + \Delta\underline{a}^I_{N_{cm,0}} \right) - M \, \underline{a}^I_{F0} \qquad \text{(A-6)} \\ &= \sum \underline{F}_{External_i} + \sum m_i \, \Delta\underline{a}^I_{N_{i,cm}} + M \, \Delta\underline{a}^I_{N_{cm,0}} - M \, \underline{a}^I_{F0} \end{aligned}$$

The $m_i \, \Delta\underline{a}^I_{N_{i,cm}}$ and $M \, \Delta\underline{a}^I_{N_{cm,0}}$ terms in (A-6) have been commonly identified as "gravitational forces" with m_i and the M multiplying $\Delta\underline{a}^I_{N_{cm,0}}$ identified as "gravitational mass". With the new formulation, however, m_i and M are inertial mass, and only appear in (A-6) by the mathematical combination of (3) and (5) with (A-5). The 0 point for position referencing is commonly selected to be an easily identified location in pure natural motion for which $\underline{a}^I_{F0}$ vanishes. An example is the center of the earth for position locations relative to the earth. For such a selection (A-6) assumes the more familiar form [3 - pp. 132]:

$$M \frac{d^2}{dt} \underline{r}^I_{cm,0} = \sum \underline{F}_{External_i} + \sum m_i \, \Delta\underline{a}^I_{N_{cm,0}} + M \, \Delta\underline{a}^I_{N_{cm,0}} \qquad \text{(A-7)}$$

APPENDIX B

TRANSLATIONAL DYNAMICS OF RIGID BODIES

For a rigid body it is more appropriate to define the m_i mass points in (A-6) as part of a continuum for which ρ_i is the body point i mass density, dv is the differential volume of mass point i, and the summation operation becomes an integral, so that (A-6) becomes

$$M \frac{d^2}{dt} \underline{r}^I_{cm,0} = \sum \underline{F}_{External_i} + \int \Delta\underline{a}^I_{N_{i,cm}} \rho_i \, dv + M \, \Delta\underline{a}^I_{N_{cm,0}} - M \, \underline{a}^I_{F_0} \qquad \text{(B-1)}$$

For rigid bodies, a very good approximation is that the $\Delta\underline{a}^I_{N_{i,cm}}$ gravity differential term in (B-1) varies linearly across the body as $\Delta\underline{a}^I_{N_{i,cm}} \equiv \nabla\left(\Delta\underline{a}^I_{N_{i,cm}}\right) \cdot \underline{r}^I_{i,cm}$ where

$\nabla\left(\Delta\underline{a}^I_{N_{i,cm}}\right)$ = Gradient of $\Delta\underline{a}^I_{N_{i,cm}}$ at the body center-of-mass cm.

Substitution into the (B-1) integral term with (A-2) for total mass M gives

$$\begin{aligned}
\int \Delta\underline{a}^I_{N_{i,cm}} \rho_i \, dv &= \nabla\left(\Delta\underline{a}^I_{N_{i,cm}}\right) \cdot \int \underline{r}^I_{i,cm} \rho_i \, dv \\
&= \nabla\left(\Delta\underline{a}^I_{N_{i,cm}}\right) \cdot \int \left(\underline{r}^I_{i,0} - \underline{r}^I_{cm,0}\right) \rho_i \, dv \\
&= \nabla\left(\Delta\underline{a}^I_{N_{i,cm}}\right) \cdot \left[\int \underline{r}^I_{i,0} \rho_i \, dv - \underline{r}^I_{cm,0} \int \rho_i \, dv \right] \\
&= \nabla\left(\Delta\underline{a}^I_{N_{i,cm}}\right) \cdot \left[\int \underline{r}^I_{i,0} \rho_i \, dv - M \, \underline{r}^I_{cm,0} \right]
\end{aligned} \qquad \text{(B-2)}$$

The integral form of (A-4) for a rigid body's center of mass is

$$\underline{r}_{cm,0} = \frac{1}{M} \int \underline{r}^I_{i,0} \rho_i \, dv \qquad \text{(B-3)}$$

With (B-3) in (B-2), (B-1) with (B-2) simplifies to the final common form for rigid body center-of-mass acceleration relative to arbitrary point 0:

$$\frac{d^2}{dt} \underline{r}^I_{cm,0} = \frac{1}{M} \sum \underline{F}_{External_i} + \Delta\underline{a}^I_{N_{cm,0}} - \underline{a}^I_{F_0} \qquad \text{(B-4)}$$

APPENDIX C

ROTATIONAL DYNAMICS OF MASS GROUPS

The angular momentum $\underline{H}_{i,0}$ of a mass point i relative to an arbitrary mass point 0 is traditionally defined in inertial I frame coordinates as [3 - pp. 132]

$$\underline{H}^{I}_{i,0} \equiv m_i\, \underline{r}^{I}_{i,0} \times \frac{d\underline{r}^{I}_{i,0}}{dt} \tag{C-1}$$

The summation of (C-1) over the group of mass points yields the I frame angular momentum $\underline{H}^{\mathbf{I}}_{0}$ of the group:

$$\underline{H}^{\mathbf{I}}_{0} \equiv \Sigma\, \underline{H}^{I}_{i,0} = \Sigma \left(m_i\, \underline{r}^{I}_{i,0} \times \frac{d\underline{r}^{I}_{i,0}}{dt} \right) \tag{C-2}$$

The rate of change of $\underline{H}^{\mathbf{I}}_{0}$ is the derivative of (C-2)

$$\frac{d\,\underline{H}^{\mathbf{I}}_{0}}{dt} = \Sigma\, m_i \left[\frac{d\,\underline{r}^{I}_{i,0}}{dt} \times \frac{d\underline{r}^{I}_{i,0}}{dt} + \underline{r}^{I}_{i,0} \times \frac{d^2\underline{r}^{I}_{i,0}}{dt} \right] = \Sigma\, m_i\, \underline{r}^{I}_{i,0} \times \frac{d^2\underline{r}^{I}_{i,0}}{dt} \tag{C-3}$$

Using revised motion equation (3) with point 2 and 1 identified respectively as points i and 0 yields:

$$\frac{d\,\underline{H}^{\mathbf{I}}_{0}}{dt} = \Sigma\, m_i\;\; \underline{r}^{I}_{i,0} \times \left(\underline{a}^{I}_{F_i} - \underline{a}^{I}_{F0} + \Delta\underline{a}^{I}_{N_{i,0}} \right) \tag{C-4}$$

Substituting (5) for $\underline{a}^{I}_{F_i}$ and $\underline{a}^{I}_{F0}$ with (A-4) for center-of-mass position $\underline{r}_{cm,0}$ definition finds

$$\begin{aligned} \frac{d\,\underline{H}^{\mathbf{I}}_{0}}{dt} &= \Sigma\left(\underline{r}^{I}_{i,0} \times \underline{F}_i \right) + \Sigma\left(m_i\, \underline{r}^{I}_{i,0} \times \Delta\underline{a}^{I}_{N_{i,0}} \right) - \Sigma\left(m_i\, \underline{r}^{I}_{i,0} \times \underline{a}^{I}_{F0} \right) \\ &= \Sigma\left(\underline{r}^{I}_{i,0} \times \underline{F}_i \right) + \Sigma\left(m_i\, \underline{r}^{I}_{i,0} \times \Delta\underline{a}^{I}_{N_{i,0}} \right) - M\, \underline{r}^{I}_{cm,0} \times \underline{a}^{I}_{F0} \end{aligned} \tag{C-5}$$

By selecting arbitrary point 0 as the center-of-mass (point cm) for the group, $\underline{r}^{I}_{cm,0}$ is zero, and (C-5) reduces to

$$\frac{d\,\underline{H}^{\mathbf{I}}_{cm}}{dt} = \Sigma\left(\underline{r}^{I}_{i,cm} \times \underline{F}_i \right) + \Sigma\left(m_i\, \underline{r}^{I}_{i,cm} \times \Delta\underline{a}^{I}_{N_{i,cm}} \right) \tag{C-6}$$

By Newton's third law, the interactive forces between adjacent mass points at any point i are equal and opposite. Hence, as for the Appendix B linear dynamic group case, the internal/interactive moment/force terms cancel at each i point and (C-6) further simplifies to the more familiar form [3 - pp. 132]

$$\frac{d\,\underline{H}^{\mathbf{I}}_{cm}}{dt} = \Sigma\left(\underline{r}^{I}_{i,cm} \times \underline{F}_{External_i}\right) + \Sigma\left(m_i\,\underline{r}^{I}_{i,cm} \times \Delta\underline{a}^{I}_{N_{i,cm}}\right) \qquad \text{(C-7)}$$

APPENDIX D

ROTATIONAL DYNAMICS OF RIGID BODIES

Equation (10) relates the rate of change of relative position between two points as viewed in inertial and rotating coordinate frames (I and A respectively). Consider that the A frame in (10) is defined to be rotating with a body relative to the I frame, and that points 1 and 2 in the body are identified as the body's center-of-mass cm and a general point i respectively:

$$\left(\dot{\underline{r}}^{A}_{i,cm}\right) = C^{A}_{I}\left(\dot{\underline{r}}^{I}_{i,cm}\right) - \underline{\omega}^{A}_{IA} \times \underline{r}^{A}_{i,cm} \qquad \text{(D-1)}$$

For a rigid body, when viewed in the A frame, the relative position vector between any two points in the body is constant, hence, $\left(\dot{\underline{r}}^{A}_{i,cm}\right) = 0$, and (D-1) with rearrangement becomes

$$\left(\dot{\underline{r}}^{I}_{i,cm}\right) = \underline{\omega}^{A}_{IA} \times \underline{r}^{I}_{i,cm} \qquad \text{(D-2)}$$

Then substituting (D-2) into general equation (C-2) finds for the rigid body's angular momentum around its center-of-mass:

$$\underline{H}^{\mathbf{I}}_{cm} = \Sigma\left[m_i\,\underline{r}^{I}_{i,cm} \times \left(\dot{\underline{r}}^{I}_{i,cm}\right)\right] = \Sigma\left[m_i\,\underline{r}^{I}_{i,cm} \times \left(\underline{\omega}^{I}_{IA} \times \underline{r}^{I}_{i,cm}\right)\right] \qquad \text{(D-3)}$$

For a rigid body it is more appropriate to define the m_i mass points in (D-3) as part of a continuum for which ρ_i is the body point i mass density, dv is the differential volume of mass point i, and the summation operation becomes an integral, so that (D-3) becomes

$$\underline{H}^{\mathbf{I}}_{cm} = J^{I}_{cm}\,\underline{\omega}^{I}_{IA} \qquad \text{(D-4)}$$

with the moment of inertia tensor J_{cm} defined in the I frame as

$$J^I_{cm} \equiv -\left[\int \left(\underline{r}^I_{i,cm}\times\right)\left(\underline{r}^I_{i,cm}\times\right)\rho_i \, dv\right] \qquad \text{(D-5)}$$

Replacing the natural motion summation with an integration operation in (C-7) provides the corresponding differential equation for $\underline{H}_{cm}$ in an inertial I frame:

$$\frac{d\,\underline{H}^{\mathbf{I}}_{cm}}{dt} = \Sigma\left(\underline{r}^I_{i,cm}\times\underline{F}^I_{External_i}\right) + \int \left(\underline{r}^I_{i,cm}\times\Delta\underline{a}^I_{N_{i,cm}}\right)\rho_i \, dv \qquad \text{(D-6)}$$

The lever arm applied force term $\underline{r}_{i,cm}\times\underline{F}_{External_i}$ in (D-6) is commonly referred to as "torque".

Equation (D-6) with (D-4) and (D-5) is the general rotational dynamics differential equation for a rigid body in an inertial I frame [3 - pp. 132-133]. As is well known, it demonstrates that in the absence of applied forces and $\Delta\underline{a}_{N_{i,cm}}$ natural acceleration gradients, the angular momentum $\underline{H}_{cm}$ will remain stationary in magnitude and direction in inertial coordinates. The $\Delta\underline{a}_{N_{i,cm}}$ term in (D-6) (primarily due to gravity gradient) is commonly small enough to be ignored compared to the $\underline{r}_{i,cm}\times\underline{F}_{External_i}$ external torque terms. However, there remain some applications where it can be an important factor in angular rate response (e.g., gravity gradient stabilization of satellites where a long boom is extended to intentionally amplify the gravity difference $\underline{r}_{i,cm}\times\underline{F}_{External_i}$ and physical separation $\underline{r}_{i,cm}$ terms). As can be verified by expansion and evaluation in an arbitrary B frame coordinate system, it is also important to know (as demonstrated in [3 - Sect. 6-4]) that for a symmetrical mass distribution, the $\int \left(\underline{r}^I_{i,cm}\times\Delta\underline{a}^I_{N_{i,cm}}\right)\rho_i \, dv$ term vanishes under the very good approximation of constant gravity gradient across the body.

The derivative of $\underline{H}_{cm}$ is also commonly expressed in terms of the $\underline{H}_{cm}$ rate of change viewed in another general B frame rotating at $\underline{\omega}_{IB}$ relative to the I frame. Define

$$\underline{H}^B_{cm} = C^B_I\,\underline{H}^I_{cm} \qquad \text{(D-7)}$$

Following the same methodology that led to (7) finds

$$\left(\dot{\underline{H}}^B_{cm}\right) = C^B_I\left(\dot{\underline{H}}^I_{cm}\right) - \underline{\omega}^B_{IB}\times\underline{H}^B_{cm} \qquad \text{(D-8)}$$

which when substituted into (D-8) shows that

$$\left(\dot{\underline{H}}_{cm}^{B}\right) = \Sigma\left(\underline{r}_{i,cm}^{B} \times \underline{F}_{External_i}^{B}\right) - \underline{\omega}_{IB}^{B} \times \underline{H}_{cm}^{B} + \int \left(\underline{r}_{i,cm}^{B} \times \Delta\underline{a}_{N_{i,cm}}^{B}\right) \rho_i \, dv \qquad \text{(D-9)}$$

The B frame projection of $\underline{H}_{cm}$ in (D-9) is evaluated from (D-7) and (D-4):

$$\underline{H}_{cm}^{B} = C_I^B \, \underline{H}_{cm}^{I} = C_I^B \, J_{cm}^I \, \underline{\omega}_{IA}^{I} = C_I^B \, J_{cm}^I \, C_B^I \, \underline{\omega}_{IA}^{B} = J_{cm}^B \, \underline{\omega}_{IA}^{B} \qquad \text{(D-10)}$$

The B frame projection of J_{cm} in (D-10) is obtained with (D-5) from the tensor similarity transformation of J_{cm}^I and $\left(\underline{r}_{i,cm}^{I} \times\right)$ onto the B frame:

$$\begin{aligned} J_{cm}^B &= C_I^B \, J_{cm}^I \, C_B^I = - C_I^B \left[\int \left(\underline{r}_{i,cm}^{I} \times\right)\left(\underline{r}_{i,cm}^{I} \times\right) \rho_i \, dv \right] C_B^I \\ &= - \left[\int C_I^B \left(\underline{r}_{i,cm}^{I} \times\right) C_B^I \, C_I^B \left(\underline{r}_{i,cm}^{I} \times\right) C_B^I \, \rho_i \, dv \right] \qquad \text{(D-11)} \\ &= - \left[\int \left(\underline{r}_{i,cm}^{B} \times\right)\left(\underline{r}_{i,cm}^{B} \times\right) \rho_i \, dv \right] \end{aligned}$$

APPENDIX E

THE ANALYTICS OF INERTIAL NAVIGATION

A typical version of (3) and (4) implemented in a terrestrial INS is derived from (13) with rotating coordinate frame B identified as frame E having its unit vectors parallel to known reference lines fixed relative to the earth (e.g., earth's rotation axis with the other axes in earth's equatorial plane at known angular orientations relative to a selected inertial coordinate frame orientation). Point 2 is defined to be within the INS, and point 1 is defined to be the center of the earth (thus having zero force acceleration). Neglecting the variation in natural motion acceleration from the universe between the INS and earth's center, and recognizing that in earth E frame coordinates, earth's angular rate relative to the I frame is constant, (13) becomes

$$\begin{aligned} \left(\dot{\underline{v}}_{INS,Erthcntr/E}^{E}\right) = {} & \underline{a}_{F_2}^{E} + \underline{a}_{N_{INS/g-Earth}}^{E} \\ & - \underline{\omega}_{IE}^{E} \times \left(\underline{\omega}_{IE}^{E} \times \underline{r}_{INS,Erthcntr}^{E}\right) - 2 \, \underline{\omega}_{IE}^{E} \times \underline{v}_{INS,Erthcntr/E}^{E} \end{aligned} \qquad \text{(E-1)}$$

where $\underline{a}_{N_{INS/g-Earth}}$ is the component of INS natural acceleration caused by earth's gravity, and is the velocity of INS point 2 relative to earth center point 1 as viewed in E frame coordinates, is defined as:

$$\underline{v}_{INS,Erthcntr/E}^{E} \equiv \left(\dot{\underline{r}}_{INS,Erthcntr/E}^{E}\right) \qquad \text{(E-2)}$$

Now define a third coordinate frame L having one of its unit vectors aligned parallel to the local vertical with the other two unit vectors perpendicular to each other and having a known orientation relative to the E frame. Then $\underline{v}_{INS,Erthcntr/E}$ in the L frame is

$$\underline{v}^{L}_{INS,Erthcntr/E} = C^{L}_{E}\,\underline{v}^{E}_{INS,Erthcntr/E} \tag{E-3}$$

$$\left(\dot{\underline{v}}^{L}_{INS,Erthcntr/E}\right) = \left(\dot{C}^{L}_{E}\right)\underline{v}^{E}_{INS,Erthcntr/E} + C^{L}_{E}\left(\dot{\underline{v}}^{E}_{INS,Erthcntr/E}\right) \tag{E-4}$$

where the C^{L}_{E} rate of change $\left(\dot{C}^{L}_{E}\right)$ is generated from $\underline{\omega}^{L}_{EL}$, the angular rate of the L frame relative to the E frame, or similar to (9):

$$\left(\dot{C}^{L}_{E}\right) = -\left(\underline{\omega}^{L}_{EL}\times\right)C^{L}_{E} \tag{E-5}$$

Substituting (E-1), (E-3), and (E-5) into (E-4) yields the classical inertial navigation equations for calculating $\underline{v}^{L}_{INS,Erthcntr/E}$ in an INS computer:

$$\begin{aligned}\left(\dot{\underline{v}}^{L}_{INS,Erthcntr/E}\right) &= \underline{a}^{L}_{F_2} + \underline{a}^{L}_{N_{INS/g-Earth}} \\ &\quad - \underline{\omega}^{L}_{IE}\times\left(\underline{\omega}^{L}_{IE}\times\underline{r}^{L}_{INS,Erthcntr}\right) - \left(\underline{\omega}^{L}_{EL} + 2\,\underline{\omega}^{L}_{IE}\right)\times\underline{v}^{L}_{INS,Erthcntr/E}\end{aligned} \tag{E-6}$$

$$\underline{v}^{L}_{INS,Erthcntr/E} = \int_0^t \left(\dot{\underline{v}}^{L}_{INS,Erthcntr/E}\right) dt + \underline{v}^{L}_{INS,Erthcntr/E_0} \tag{E-7}$$

with

$$\begin{gathered}\underline{\omega}^{L}_{IE} = C^{L}_{E}\,\underline{\omega}^{E}_{IE} \qquad \underline{r}^{L}_{INS,Erthcntr} = C^{L}_{E}\,\underline{r}^{E}_{INS,Erthcntr} \qquad \underline{\omega}^{L}_{EL} = F_c\,\underline{v}^{L}_{INS,Erthcntr/E} \\ \underline{r}^{E}_{INS,Erthcntr} = \text{Function of } C^{L}_{E} \text{ and } h \qquad h = \int_0^t \underline{u}_{Up}\cdot\underline{v}^{L}_{INS,Erthcntr/E}\,dt + h_0\end{gathered} \tag{E-8}$$

where

h = Altitude above earth's surface.

$\underline{u}_{Up}$ = Unit vector upward at the local position.

F_c = Curvature matrix [4 - pp. 5-21 & 5-22] (an analytic function of C^{L}_{E} and h), and approximately equal to identity divided by the sum of earth's average radius plus h.

In typical practice, C^{L}_{E} in (E-8)is calculated as the transpose of the integral of transposed (E-5), noting from its definition, that the transpose of the skew symmetric cross-product matrix form of a vector equals its negative i.e., $\left(\underline{\omega}^{L}_{EL}\times\right)^{T} = -\left(\underline{\omega}^{L}_{EL}\times\right)$:

$$\left(\dot{C}_L^E\right) = C_L^E\left(\underline{\omega}_{EL}^L \times\right) \qquad \text{(E-9)}$$

For a strapdown INS, the inertial sensors are mounted to a platform that is fixed relative to vehicle axes. Strapdown platform coordinates ("body" B frame) are defined as having unit vectors parallel to sensor platform axes. The L frame force components of $\underline{a}_{F_2}^L$ in (E-6) are obtained from the B frame force acceleration measurements $\underline{a}_{F_2}^B$ by transforming to the L frame:

$$\underline{a}_{F_2}^L = C_B^L\, \underline{a}_{F_2}^B \qquad \text{(E-10)}$$

With $\underline{v}_{INS,Erthcntr/E}^L$ found from (E-7), the relative position vector $\underline{r}_{INS,Erthcntr}^L$ in (E-8) can be found by transforming $\underline{v}_{INS,Erthcntr/E}^L$ to the E frame and integrating (E-2). In practice, however, position relative to the earth is usually measured by altitude h above the earth's surface and by angular units over the earth's surface (i.e., latitude and longitude). Latitude and longitude can be extracted from the C_L^E matrix calculated in (E-6) [4 - pp. 4-29]; altitude is determined as the integral of the vertical component of $\underline{v}_{INS,Erthcntr/E}^L$ from (E-7). The C_B^L matrix in (E-10) is calculated by integrating classic attitude rate equation forms (e.g., direction cosines - [4 - pp. 3-54]) using strapdown gyro measurements of $\underline{\omega}_{IB}^B$, the B frame angular rate relative to I frame inertial coordinates:

$$\left(\dot{C}_B^L\right) = C_B^L\left(\underline{\omega}_{IB}^B \times\right) - \left(\underline{\omega}_{IL}^L \times\right) C_B^L \qquad \text{(E-11)}$$

with

$$\underline{\omega}_{IL}^L = \underline{\omega}_{IE}^L + \underline{\omega}_{EL}^L \qquad \text{(E-12)}$$

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