INTRODUCTION TO THE KINEMATICS OF POINT-TO-POINT RELATIVITY

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ABSTRACT

Point-To-Point Relativity is a revised form of traditional Relativity theory in which position is described as the distance vector between two points in space as viewed by observers translating relative to one-another. Unlike traditional Relativity theory, the Point-to-Point approach avoids the use of relatively translating coordinate frames, space-time diagrams, world lines intersecting with space-time events, and the concept of space-time simultaneity. In the Point-to-Point approach, distance vectors are represented as free vectors having no preferred location in coordinate frames in which they are described, and coordinate frames are used only as angular references for projecting vector quantities along their axes (as the dot product with mutually orthogonal coordinate frame unit vectors). This article describes the Point-to-Point kinematic approach, deriving Point-to-Point Lorentz relationships between observers travelling relative to one-another and from this, analytically demonstrating Lorentz time dilation, distance contraction, and invariant proper time. Overall results match their equivalents obtained using traditional Relativity theory. As part of the Point-to-Point formulation, a new notation is developed to explicitly identify point-to-point distance-vectors/time-intervals measured by a particular observer, and their relationship with equivalent measurements taken by another observer.

1.0 INTRODUCTION

Newton's fundamental laws of motion state that the velocity of one spatial point relative to another is of the same magnitude but oppositely directed from the relative velocity of the second point relative to the first [1, pp. 416]. Einstein's theory of Relativity generalized the Newtonian concept, stating that general laws of motion are the same for any observer [2 Chpt. 5, 3 pp. 177]. A key element in Relativity theory is the early findings (e.g., Michelson and Morley [4]) that the velocity of electro-magnetic (e/m) radiation (i.e., the "velocity of light") through space is the same when measured by any observer, regardless of whether the e/m emission source is in motion relative to the observer. Theoretical investigations by Lorentz in1895 [5, Sect. II) used this principle and Maxwell's electro-magnetic equations to describe the propagation of e/m waves in space as viewed by two observers in motion relative to one another. A key result of these investigations was the Lorentz Transformation formulas relating distance and time measurements between the observers. Among other significant findings, Einstein's theory of Relativity leads to the same transformation formulas developed by Lorentz.

Both traditional Newtonian and Relativity theory describe relative motion in classical Cartesian coordinate frames that translate relative to one-another, with relative motion reported by observers translating with the coordinate frames. In contrast, Point-to-Point kinematics describes relative motion in terms of the distance vector between two points in space as measured at two observation points translating relative to one-another. Coordinate frames are only used as devices for numerically evaluating orthogonal components of relative motion vector parameters (position, velocity, acceleration). As such, a Point-to-Point coordinate frame is represented by three orthogonal unit free vectors having no specific coordinate origin, with Point-to-Point vector components "along" coordinate frame axes numerically calculated as the dot product between the vector and the coordinate frame unit vectors.

Point-to-Point kinematic theory was formally introduced in [6] based on a modified version of Newtonian dynamics. For compatibility with Relativity theory, this article expands on the Point-to-Point kinematic approach so that observations of e/m wave propagation through space (at the "speed-of-light") is the same for any observer. As with traditional Relativity theory, the expanded Point-to-Point approach evolves into the requirement that both time and distance become interrelated parameters, dependent on the relative motion between observers. Thus, Point-to-Point kinematics generate equivalent conclusions obtained by traditional Relativity theory relating kinematic parameters measured by observers that are in motion relative to one-another; e.g., time dilation and distance contraction of comparative events, "proper time" invariance between observers, and the equivalent to Lorentz transformation operations.

Formulations of traditional Relativity theory have typically included a hypothetical means to synchronize clocks between separated observers so that observed spatial events can be compared at common instants of time [2 Chpt. 8, 7 Sect. 12-2, 8 Chpt. VI Sect. 1]. This has typically involved a hypothesized clock measurement e/m message transfer procedure between observers while observing remote identifiable spatial events (or an instant of time when the separated observers are at the same spatial location). Because the Point-to-Point approach is based on the distance between two remote events during the <u>time interval</u> between event occurrences, the need to account for clock <u>time</u> synchronization between observers is avoided.

Point-to-Point kinematics is presented in this article in vector format for vector component evaluation in "non-rotating inertial space", i.e., coordinate frames in which traditional Newtonian and Relativity theory have been defined to be valid. Appendix A provides a measurable definition of non-rotating inertial coordinates based on the redefined Newtonian motion formulation in [6]. It will be assumed in this article that the Appendix A inertial space definition also applies for Relativity theory in general. Planned future articles will expand on the results of this article for compatibility with projection on coordinate frames that are rotating relative to the Appendix A defined inertial space.

As has been past practice in simplified derivations of traditional Relativity formulas, this article is based on a constant relative velocity between observers in non-rotating inertial space. Planned future articles will expand on these results to account for changing velocity between observers during the time interval that remote events are observed.

This article begins with a generalized analytical description of Point-to-Point kinematics in terms of the relative distance vector between two event points in space determined at two observation points, first for compatibility with Newtonian kinematic theory, then for compatibility with Relativity theory. The result for Point-to-Point Relativity is the equivalent of the Lorentz transformation in traditional Relativity theory relating remote event distance measurements at one observation point to those at another that is in motion relative to the first. This article also analyzes degraded versions of the two-event/two-observer case; for two events occurring at two different times at a single point, and for two events occurring at two different times at one of the observation points.

Based on the Lorentz transformation equivalency results, this article then demonstrates the Point-to-Point equivalent to Lorentz time dilation, length contraction, and proper time invariance, the latter in terms of infinitesimal distance changes over an infinitesimal time interval. The article concludes with a derivation of the Point-to-Point equivalent to Lorentz velocity and acceleration transformation of remote points in relative motion as viewed by observers in motion relative to one-another.

2.0 TERMINOLOGY

2.1 Space-Time, Events, Observations, And Transformations

For those familiar with inertial navigation, "space-time" in traditional Relativity parlance is a way of describing position locations at particular time points (commonly referred to as "time stamps"). Thus, position with a time stamp is what has been defined as a location in four-dimensional "space-time", three being the traditional spatial position dimensions, the fourth being time. An "event" in traditional Relativity theory is something that occurs instantaneously at a specific point in space-time [2 pp. 36, 7 pp. 515, 8 pp. 28]. Examples of events typically used in deriving Relativity equations are lightening strikes, reflected light signals, and reflections from radar transmissions [2 pp. 29, 7 pp. 521, 9 pp. 10]. This article will use similar terminology, but with the general understanding that an "event" analytically represents a particular location/time-instant, regardless of the example used to characterize it. Symbols p and q will be used to identify event point locations, a and b will be used to identify location points where the events are observed, with points a and b being in motion relative to each other. In some examples, an event may be defined to occur at one of the observation points at the event time instant, thereby classifying the observation point as an event at that instant of time.

In traditional Relativity, "transformation" refers to a change in data measurements from one coordinate frame to another, both in linear motion relative to the other (e.g., the "Lorentz transformation" which is referred to extensively in this article). The Point-to-Point Relativity equivalent to Lorentz transformation is referred to in this article as Lorentz "conversion". The word "transformation" will refer to transforming vector data from its component form in one coordinate frame to its component form in another, both at angular orientation relative to the other. Those familiar with modern-day inertial navigation will recognize this as the process used in "transforming" accelerometer data measured in inertial sensor assembly coordinates (parallel to user vehicle axes) to their equivalent form in navigation coordinates (e.g., a locally level reference frame).

2.2 Basic Notation

For compatibility with [6], the following basic notation is used in this article to describe spatial distance and time parameters measured by observers at particular spatial position locations. For convenience to the reader, the definitions are repeated where they are first introduced, as are variations and other definitions introduced in the main text.

$\underline{x}_{i\,j}$ = Distance vector between spatial points *i* and *j* at an arbitrary time instant. Appendix C shows how the distance vector from point *i* to *j* can be ascertained by an observer at point *i* using electro-magnetic (e/m) time of travel measurements from point *i* to point *j*, then reflected back to point *i*.

Time instants 1 and 2 = Particular time instants when measurable events occur, at point *p* at time instant 1, and another at point *q* at time instant 2 (e.g., when radar transmitted e/m pulses were reflected from points *p* and *q* being tracked). Time instant 2 is defined to be later than time instant 1. An example is provided in Appendix D, showing how the time instant for an event can be ascertained on a clock located where the event is being observed.

$\underline{x}_{i\,j,1}$, $\underline{x}_{i\,j,2}$ = $\underline{x}_{i\,j}$ at time instants 1 and 2.

$\underline{x}_{i\,p_1/i}$, $\underline{x}_{i\,q_2/i}$ = Measured distance vector from observation point *i* to event point *p* at event time instant 1, and from point *i* to event point *q* at event time instant 2, both measured at point *i* (*i* = *a* or *b*).

$\underline{x}_{p_1\,q_2/i}$ = Distance vector from event point *p* at time instant 1 to event point *q* at time instant 2 as calculated at point *i* (*i* = *a* or *b*) from distance vector measurements taken at points *a* and *b* to event points *p* and *q*.

$\underline{x}_{ab,1/a}$, $\underline{x}_{ab,2/a}$ = Observer *a* measured distance vector (/*a* notation) from observation point *a* to observation point *b* at event time instants 1 and 2.

$\Delta\underline{x}_{ab,1\to 2/a}$ = Change in the distance vector from point *a* to point *b* during the time interval from event time instant 1 to 2 as determined at point *a* from $\underline{x}_{ab,1/a}$ and $\underline{x}_{ab,2/a}$ measurements.

$\underline{x}_{ba,1/b}$, $\underline{x}_{ba,2/b}$ = Observer *b* measured distance vector (/*b* notation) from observation point *b* to observation point *a* at event time instants 1 and 2.

$\Delta \underline{x}_{ba,1\to 2/b}$ = Change in the distance vector from point b to point a during the time interval from event time instant 1 to 2 as determined at point b from $\underline{x}_{ba,1/b}$ and $\underline{x}_{ba,2/b}$ measurements.

$\underline{x}_{b_1 p_1/a}$, $\underline{x}_{b_2 q_2/a}$ = Distance vector from observation point b to event point p at time instant 1, and from observation point b to event point q at time instant 2, both estimated at observation point a based on the calculated values of $\underline{x}_{b\,p_1/b}$ and $\underline{x}_{b\,q_2/b}$. The estimation method depends on whether the basis is Newtonian or Relativity theory.

$\underline{x}_{a_1 p_1/b}$, $\underline{x}_{a_2 q_2/b}$ = Distance vector from observation point a to event point p at time instant 1 and from observation point a to event point q at time instant 2, both estimated at observation point b based on the calculated values of $\underline{x}_{a\,p_1/b}$ and $\underline{x}_{a\,q_2/b}$. The estimation method depends on whether the basis is Newtonian or Relativity theory.

$t_{a,1}$, $t_{a,2}$, $t_{b,1}$, $t_{b,2}$ = Event time instants 1 and 2 (1 occurring before 2) registered on clocks located at observation points a and b.

$\Delta t_{a,1\to 2}$, $\Delta t_{b,1\to 2}$ = The time interval between event time instants 1 and 2 elapsed on the observation point a and b clocks.

$\underline{v}_{ab/a}$ = Rate of change of $\underline{x}_{ab/a}$ measured using a point a clock (i.e., velocity of point b relative to point a as measured at point a).

$\underline{v}_{ba/b}$ = Rate of change of $\underline{x}_{ba/b}$ measured using a point b clock (i.e., velocity of point a relative to point b as measured at point b).

$\underline{v}_{ab/b}$ = The negative of $\underline{v}_{ba/b}$ (also shown to be $\underline{v}_{ab/a}$, defined previously).

$\underline{u}_{\upsilon}$ = Unit vector parallel to $\underline{v}_{ab/a}$.

v_{ab} = Magnitude of $\underline{v}_{ab/a}$ (and of $\underline{v}_{ba/b}$).

$\underline{x}_{p_1\,q_2/i_{\perp}}$ = Component of $\underline{x}_{p_1\,q_2/i}$ perpendicular to $\underline{u}_{\upsilon}$ ($i = a$ or b).

3.0 POINT-TO-POINT KINEMATIC FUNDAMENTALS

Consider two events in space-time, one occurring at spatial point *p* at time instant 1 (call it space-time point p_1), the other at spatial point *q* at time instant 2 (call it space-time point q_2). Consider two observers, one at point *a*, the other at point *b*, and that the observers can independently calculate the distance vector from their location to the p_1 and q_2 events (e.g., as in the Appendix D example):

$$\underline{x}_{p_1 q_2/a} = \underline{x}_{a q_2/a} - \underline{x}_{a p_1/a} \qquad \underline{x}_{p_1 q_2/b} = \underline{x}_{b q_2/b} - \underline{x}_{b p_1/b} \tag{1}$$

where

$\underline{x}_{a p_1/a}$, $\underline{x}_{a q_2/a}$ = Distance vectors from point *a* to point *p* at time instant 1, and from point *a* to point *q* at time instant 2, both measured at point *a*.

$\underline{x}_{b p_1/b}$, $\underline{x}_{b q_2/b}$ = Distance vectors from point *b* to point *p* at time instant 1, and from point *b* to point *q* at time instant 2, both measured at point *b*.

$\underline{x}_{p_1 q_2/a}$, $\underline{x}_{p_1 q_2/b}$ = Distance vectors from point *p* at time instant 1 to point *q* at time instant 2 determined from the points *a* and *b* measurements.

Figs. 1a and 1b illustrate the Eq. (1) geometry from the perspective of the point *a* and *b* observers.

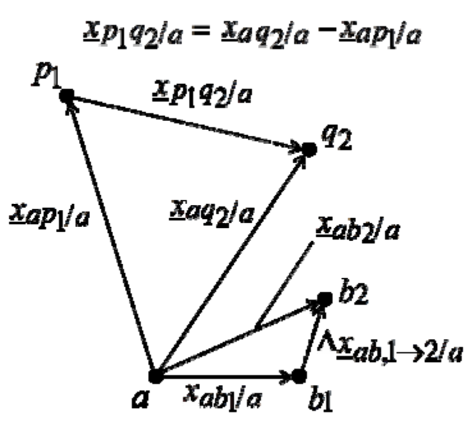

Fig. 1a - Distance Vectors From Observer *a* Viewpoint

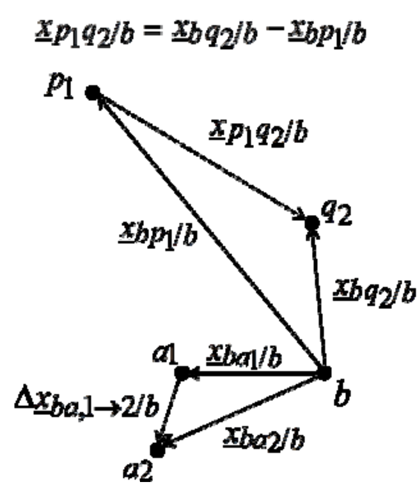

Fig. 1b - Distance Vectors From Observer *b* Viewpoint

In Figs. 1a and 1b, p_1 and q_2 refer to spatial points *p* and *q* at event time instants 1 and 2. In Fig. 1a, b_1 and b_2 refer to the location of observation point *b* at event time instants 1 and 2. In

Fig. 1b, a_1 and a_2 refer to the location of observation point *a* at event time instants 1 and 2. Included in the figures (for discussion to follow) are distance vectors from each observer to the other at time instants 1 and 2 ($\underline{x}_{ab1/a}$, $\underline{x}_{ab2/a}$, $\underline{x}_{ba1/b}$, $\underline{x}_{ba2/b}$), and the change in each observer's $\underline{x}_{ab/a}$ measurements over the time instant 1 to 2 interval ($\Delta\underline{x}_{ab,1\to 2/a}, \Delta\underline{x}_{ba,1\to 2/b}$). Distance vectors that are directly measureable by each observer (as discussed previously) are shown as solid lines. Dotted lines identify vectors that are calculated from the measurable vectors.

Note, that because there is no preferred observation point in Newtonian or Relativity theory, a vector from point *a* to point *b* at a particular time instant will be equal in magnitude, but oppositely directed, from a vector from point *b* to point *a* at the same time instant. Thus, the $\underline{x}_{ab,1/a}$ vector from observation point *a* to p_1 in Fig. 1a is equal in magnitude but oppositely directed from *b* to a_1 distance vector $\underline{x}_{ba,1/b}$ in Fig. 1b. Similarly, the $\underline{x}_{ab,2/a}$ vector from observation point *a* to b_2 in Fig. 1a, is equal in magnitude but oppositely directed from *b* to a_2 distance vector $\underline{x}_{ba,2/b}$ in Fig. 1b.

Using the distance vector measurements between *p* and *q* events determined in (1), and the distance vectors between observers shown in Figs. 1a and 1b, each observer can deduce what the other should find for the distance vector between the *p* and *q* events. First, we express the $\underline{x}_{aq_2/a}$, $\underline{x}_{ap_1/a}$, $\underline{x}_{bq_2/b}$, and $\underline{x}_{bp_1/b}$ measured terms in (1) as functions of measureable distances between *a* and *b* and observer immeasurable parameters $\underline{x}_{b_2q_2/a}$, $\underline{x}_{b_1p_1/a}$, $\underline{x}_{a_2q_2/b}$, $\underline{x}_{a_1p_1/b}$:

$$\begin{aligned} \underline{x}_{aq_2/a} &= \underline{x}_{b_2q_2/a} + \underline{x}_{ab,2/a} \qquad & \underline{x}_{ap_1/a} &= \underline{x}_{b_1p_1/a} + \underline{x}_{ab,1/a} \\ \underline{x}_{bq_2/b} &= \underline{x}_{a_2q_2/b} + \underline{x}_{ba,2/b} \qquad & \underline{x}_{bp_1/b} &= \underline{x}_{a_1p_1/b} + \underline{x}_{ba,1/b} \end{aligned} \tag{2}$$

To illustrate the geometrical relationships between the Eqs. (2) parameters, portions of Figs. 1a and 1b have been redrawn in Figs. 2a and 2b, but including the $\underline{x}_{b_2q_2/a}$, $\underline{x}_{b_1p_1/a}$, $\underline{x}_{a_2q_2/b}$, and $\underline{x}_{a_1p_1/b}$ observer immeasurable vectors (shown dotted, as are the $\underline{x}_{b_2q_2/a}$ and $\underline{x}_{a_2q_2/b}$ immeasurables brought over from Figs. 1a and 1b). In Figs. 2a, 2b, and in Eqs. (2):

$\underline{x}_{b_1p_1/a}$, $\underline{x}_{b_2q_2/a}$ = Distance vectors from point *b* to event points *p* and *q* at event time instants 1 and 2, both as determined at observation point *a*.

$\underline{x}_{a_1p_1/b}$, $\underline{x}_{a_2q_2/b}$ = Distance vectors from point *a* to event points *p* and *q* at event time instants 1 and 2, both as determined at observation point *b*.

$\underline{x}_{ab,1/a}$, $\underline{x}_{ab,2/a}$, $\underline{x}_{ba,1/b}$, $\underline{x}_{ba,2/b}$ = Measurements at points *a* and *b* of the distance vector between the two points at time instants 1 and 2.

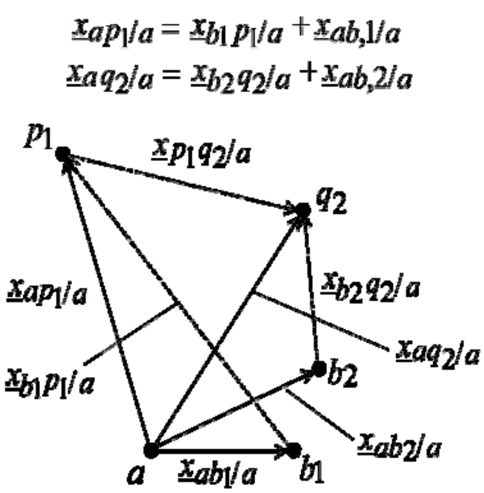

Fig. 2a - Construction of $\underline{x}_{b_2q_2/a}$
And $\underline{x}_{b_1p_1/a}$ By Observer a

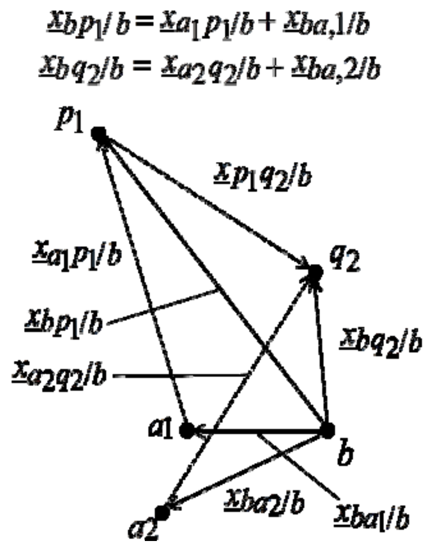

Fig. 2b - Construction of $\underline{x}_{a_2q_2/b}$
And $\underline{x}_{a_1p_1/b}$ By Observer b

Substituting (2) in (1) obtains

$$
\begin{aligned}
\underline{x}_{p_1q_2/a} &= \underline{x}_{b_2q_2/a} + \underline{x}_{ab,2/a} - \underline{x}_{b_1p_1/a} - \underline{x}_{ab,1/a} = \underline{x}_{b_2q_2/a} - \underline{x}_{b_1p_1/a} + \Delta\underline{x}_{ab,1\to2/a} \\
\underline{x}_{p_1q_2/b} &= \underline{x}_{a_2q_2/b} + \underline{x}_{ba,2/b} - \underline{x}_{a_1p_1/b} - \underline{x}_{ba,1/b} = \underline{x}_{a_2q_2/b} - \underline{x}_{a_1p_1/b} + \Delta\underline{x}_{ba,1\to2/b}
\end{aligned}
\tag{3}
$$

in which, as in Figs. 1a and 1b,

$$
\Delta\underline{x}_{ab,1\to2/a} \equiv \underline{x}_{ab,2/a} - \underline{x}_{ab,1/a} \qquad \Delta\underline{x}_{ba,1\to2/b} \equiv \underline{x}_{ba,2/b} - \underline{x}_{ba,1/b} \tag{4}
$$

where

$\Delta\underline{x}_{ab,1\to2/a}$, $\Delta\underline{x}_{ba,1\to2/b}$ = Change in the distance vectors between points a and b during time instants 1 and 2 as determined at points a and b.

Note that $\underline{x}_{b_1p_1/a}$ and $\underline{x}_{b_2q_2/a}$ in Fig. 2a appear equal to $\underline{x}_{bp_1/b}$ and $\underline{x}_{bq_2/b}$ in Fig. 2b. Also, that $\underline{x}_{a_1p_1/b}$ and $\underline{x}_{a_2q_2/b}$ in Fig. 2b appear equal to $\underline{x}_{ap_1/a}$ and $\underline{x}_{aq_2/a}$ in Fig. 2a. That is because Figs. 1a, 1b and 2a, 2b were constructed from Eqs. (1) and (2) assuming these quantities would be equal, whether calculated or measured in the a or the b frames. In Newtonian kinematic theory this would exactly be true. In Relativity theory, however, it is only approximately true, the accuracy depending on the relative motion between points a and b during time instants 1 and 2.

3.1 Newtonian Formulation

In Newtonian kinematic theory, distance vectors between two points are the same whether determined or calculated at point *a* or point *b*. Thus, referring to Figs. 2a and 2b:

Based on Newtonian kinematic theory:

$$\underline{x}_{b_1 p_1/a} = \underline{x}_{bp_1/b} \qquad \underline{x}_{b_2 q_2/a} = \underline{x}_{b q_2/b} \qquad \underline{x}_{a_1 p_1/b} = \underline{x}_{ap_1/a} \qquad \underline{x}_{a_2 q_2/b} = \underline{x}_{aq_2/a} \tag{5}$$

Substituting (5) into (3) obtains

$$\begin{aligned} \underline{x}_{p_1 q_2/a} &= \underline{x}_{b q_2/b} - \underline{x}_{bp_1/b} + \Delta\underline{x}_{ab,1\to 2/a} = \underline{x}_{p_1 q_2/b} + \Delta\underline{x}_{ab,1\to 2/a} \\ \underline{x}_{p_1 q_2/b} &= \underline{x}_{aq_2/a} - \underline{x}_{ap_1/a} + \Delta\underline{x}_{ba,1\to 2/b} = \underline{x}_{p_1 q_2/a} + \Delta\underline{x}_{ba,1\to 2/b} \end{aligned} \tag{6}$$

$$\underline{x}_{p_1 q_2/b} = \underline{x}_{p_1 q_2/a} - \Delta\underline{x}_{ab,1\to 2/a} \qquad \underline{x}_{p_1 q_2/a} = \underline{x}_{p_1 q_2/b} - \Delta\underline{x}_{ba,1\to 2/b} \tag{7}$$

Figs. 3a and 3b illustrate the construction of $\underline{x}_{p_1 q_2/b}$ and $\underline{x}_{p_1 q_2/a}$ in (7) and their comparison with $\underline{x}_{p_1 q_2/b}$ and $\underline{x}_{p_1 q_2/a}$ from Figs. 1a and 1b.

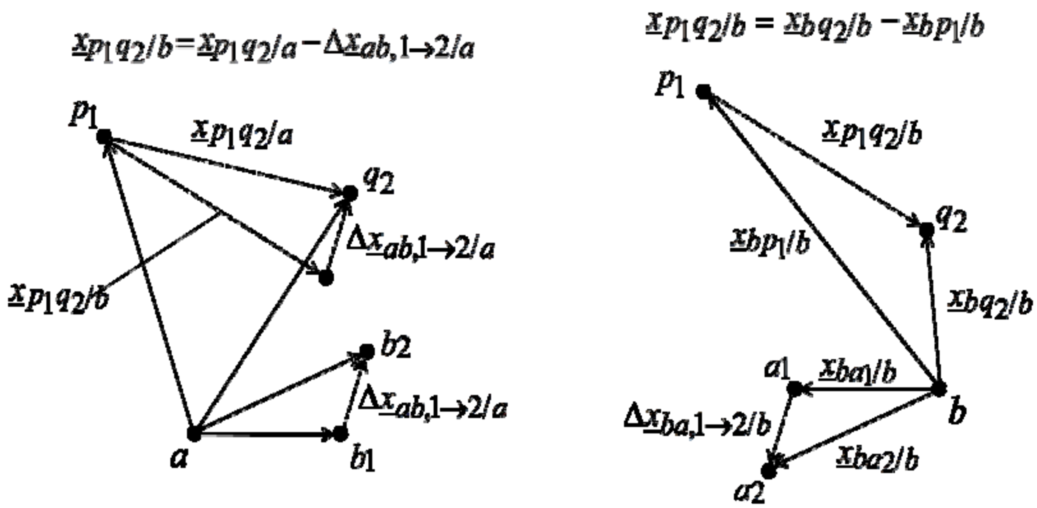

Fig. 3a - Point *a* Vector Construction of $\underline{x}_{p_1 q_2/b}$ (On Left) Compared With The Point *b* Construction Of $\underline{x}_{p_1 q_2/b}$ From Fig. 1b (On Right)

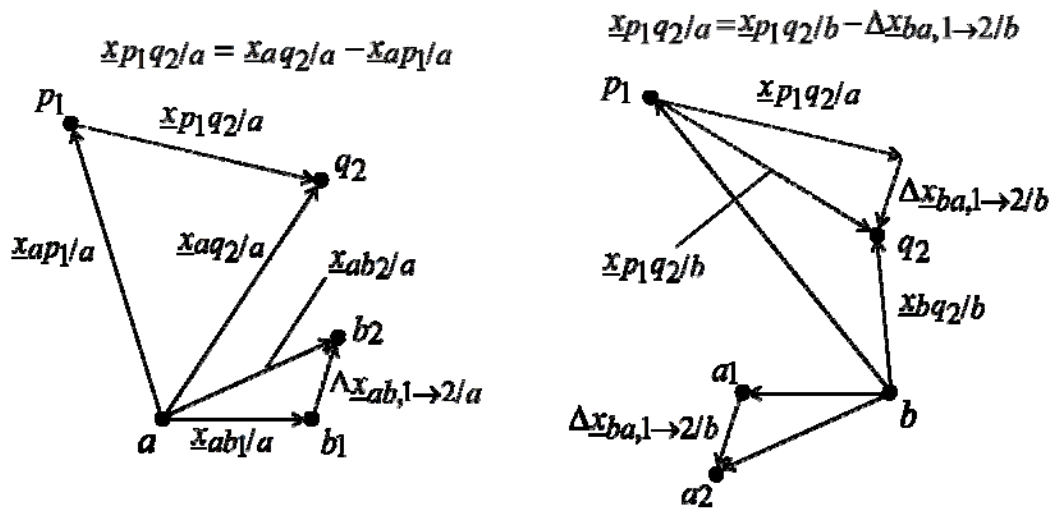

Fig. 3b - Point *b* Vector Construction of $\underline{x}_{p_1 q_2/a}$ (On Right) Compared With The Point *a* Construction Of $\underline{x}_{p_1 q_2/a}$ From Fig. 1a (On Left)

The $\Delta\underline{x}_{ab,1\rightarrow 2/a}$ and $\Delta\underline{x}_{ba,1\rightarrow 2/b}$ terms in (6) and (7) can be expressed as functions of the relative velocity between points *a* and *b* during the interval between time instants 1 and 2. From both traditional Newtonian and Relativity theory, the relative velocities between points *a* and *b* are equal and opposite with respect to each other's time reference:

$$\underline{v}_{ab/a} \equiv \frac{d}{dt_a}\underline{x}_{ab/a} \qquad \underline{v}_{ba/b} \equiv \frac{d}{dt_b}\underline{x}_{ba/b} \qquad \underline{v}_{ba/b} = -\,\underline{v}_{ab/a} \tag{8}$$

where

$\underline{v}_{ab/a}$ = Rate of change of $\underline{x}_{ab/a}$ as measured on the point *a* clock (i.e., velocity of point *b* relative to point *a* as measured at point *a*).

$\underline{v}_{ba/b}$ = Rate of change of $\underline{x}_{ba/b}$ as measured on the point *b* clock (i.e., velocity of point *a* relative to point *b* as measured at point *b*).

Figs. 4a and 4b illustrate the Equation (8) velocity vectors from the perspective of the point *a* and *b* observers.

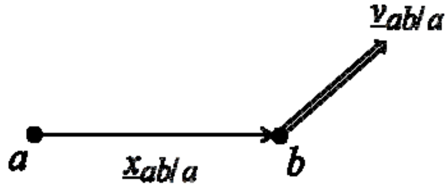

Fig. 4a - Relative Velocity Vector From Observer *a* Viewpoint

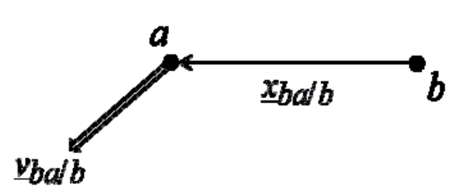

Fig. 4b - Relative Velocity Vector From Observer *b* Viewpoint

We can also define:

$$v_{ab} \equiv \left|\underline{v}_{ab/a}\right| \qquad \underline{u}_{\upsilon} \equiv \underline{v}_{ab/a} / v_{ab} \tag{9}$$

so that from (8),

$$\underline{v}_{ab/a} = v_{ab}\,\underline{u}_{\upsilon} \qquad \underline{v}_{ba/b} = -\,\underline{v}_{ab/a} = -\,v_{ab}\,\underline{u}_{\upsilon} \qquad \left|\underline{v}_{ba/b}\right| = v_{ab} \tag{10}$$

where

$\underline{u}_{\upsilon}$ = Unit vector parallel to $\underline{v}_{ab/a}$.

v_{ab} = Magnitude of $\underline{v}_{ab/a}$ (and of $\underline{v}_{ba/b}$).

It is to be noted that from the basic definition of distance vectors between spatial points:

$$\underline{x}_{ba/b} = -\,\underline{x}_{ab/b} \tag{11}$$

where

$\underline{x}_{ab/b}$ = Distance vector from point *a* to point *b* from the viewpoint of observer *b*.

Then (for future use), based on $\underline{v}_{ba/b} = -\,\underline{v}_{ab/a}$ in (8), the $\underline{v}_{ba/b}$ definition following (8), and with (11):

$$\underline{v}_{ab/a} = -\,\underline{v}_{ba/b} = -\frac{d}{dt_b}\underline{x}_{ba/b} = \frac{d}{dt_b}\underline{x}_{ab/b} \equiv \underline{v}_{ab/b} \tag{12}$$

where

$\underline{v}_{ab/b}$ = Rate of change of $\underline{x}_{ab/b}$ (the negative of $\underline{x}_{ba/b}$) as measured on the point *b* clock (i.e., velocity of point *b* relative to point *a* as measured at point *b*).

Per traditional introductory Relativity analytical design procedures, we now assume (and for the remainder of this article) that the relative velocity between points *a* and *b* will be constant. Then during the time interval between *p* and *q* event times 1 and 2, the $\Delta\underline{x}_{ab,1\to 2/a}$ and $\Delta\underline{x}_{ba,1\to 2/b}$ distance vector changes in (6) and (7) will be from (8) and (10):

$$\begin{aligned} \Delta\underline{x}_{ab,1\to 2/a} &= \underline{v}_{ab/a}\,\Delta t_{a,1\to 2} = v_{ab}\,\Delta t_{a,1\to 2}\,\underline{u}_{\upsilon} \\ \Delta\underline{x}_{ba,1\to 2/b} &= \underline{v}_{ba/b}\,\Delta t_{b,1\to 2} = -\,v_{ab}\,\Delta t_{b,1\to 2}\,\underline{u}_{\upsilon} \end{aligned} \tag{13}$$

in which

$$\Delta t_{a,1\to 2} \equiv t_{a,2} - t_{a,1} \qquad \Delta t_{b,1\to 2} \equiv t_{b,2} - t_{b,1} \tag{14}$$

where

$\Delta t_{a,1\to 2}$, $\Delta t_{b,1\to 2}$ = Time interval time between p and q event time instants 1 and 2 as measured on the point a and b clocks.

With (13), (7) becomes the classical Newtonian form [2 pp. 37, 7 pp. 508, 8 pp. 237, 9 pp.19]:

$$\begin{aligned} &\underline{x}_{p_1 q_2/b} = \underline{x}_{p_1 q_2/a} - v_{ab}\, \Delta t_{a,1\to 2}\, \underline{u}_v \qquad \Delta t_{b,1\to 2} = \Delta t_{a,1\to 2} \\ &\underline{x}_{p_1 q_2/a} = \underline{x}_{p_1 q_2/b} + v_{ab}\, \Delta t_{b,1\to 2}\, \underline{u}_v \qquad \Delta t_{a,1\to 2} = \Delta t_{b,1\to 2} \end{aligned} \tag{15}$$

Included in Eqs. (15) is the Newtonian kinematic assumption of equality between time intervals measured at points a or b.

3.2 General Formulation For Newtonian And Relativity Compatibility

To develop a general formulation that is compatible with either Newtonian and Relativity theory, means must be introduced in (15) to account for constancy in speed-of-light measurements when specializing to Relativity kinematics. For the development, we first decompose the Eqs. (1) $\underline{x}_{p_1 q_2/a}$, $\underline{x}_{p_1 q_2/b}$ computed vectors and the Eqs. (3) $\underline{x}_{ap_1/a}$, $\underline{x}_{aq_2/a}$, $\underline{x}_{bp_1/b}$, $\underline{x}_{bq_2/b}$ measurable vectors, into components parallel and perpendicular to $\underline{u}_v$:

$$\underline{x}_{p_1 q_2/a} = \underline{x}_{p_1 q_2/a} \cdot \underline{u}_v\, \underline{u}_v + \underline{x}_{p_1 q_2/a_\perp} \qquad \underline{x}_{p_1 q_2/b} = \underline{x}_{p_1 q_2/b} \cdot \underline{u}_v\, \underline{u}_v + \underline{x}_{p_1 q_2/b_\perp} \tag{16}$$

$$\begin{aligned} &\underline{x}_{ap_1/a} = \underline{x}_{ap_1/a} \cdot \underline{u}_v\, \underline{u}_v + \underline{x}_{ap_1/a_\perp} \qquad \underline{x}_{aq_2/a} = \underline{x}_{aq_2/a} \cdot \underline{u}_v\, \underline{u}_v + \underline{x}_{aq_2/a_\perp} \\ &\underline{x}_{bp_1/b} = \underline{x}_{bp_1/b} \cdot \underline{u}_v\, \underline{u}_v + \underline{x}_{bp_1/b_\perp} \qquad \underline{x}_{bq_2/b} = \underline{x}_{bq_2/b} \cdot \underline{u}_v\, \underline{u}_v + \underline{x}_{bq_2/b_\perp} \end{aligned} \tag{17}$$

where

$\underline{x}_{p_1 q_2/a_\perp}$, $\underline{x}_{p_1 q_2/b_\perp}$ = Components of $\underline{x}_{p_1 q_2/a}$ and $\underline{x}_{p_1 q_2/b}$ perpendicular to $\underline{u}_v$.

$\underline{x}_{ap_1/a_\perp}$, $\underline{x}_{aq_2/a_\perp}$, $\underline{x}_{bp_1/b_\perp}$, $\underline{x}_{bq_2/b_\perp}$ = Components of $\underline{x}_{ap_1/a}$, $\underline{x}_{aq_2/a}$, $\underline{x}_{bp_1/b}$, $\underline{x}_{bq_2/b}$ perpendicular to $\underline{u}_v$.

Based on traditional Relativity theory, the components of either $\underline{x}_{ap_1/a}$, $\underline{x}_{aq_2/a}$, $\underline{x}_{bp_1/b}$ or $\underline{x}_{bq_2/b}$ parallel and perpendicular to v_{ab} motion will be independent from one another, and the components perpendicular to $\underline{u}_v$ will be independent of v_{ab} induced Relativistic effects (i.e., behave in classical (5) Newtonian fashion). Thus, similar to [8 pp. 236], observer a finds using $\underline{x}_{b1\,p_1/a_\perp} = \underline{x}_{b\,p_1/b_\perp}$ and $\underline{x}_{b2\,q_2/a_\perp} = \underline{x}_{b\,q_2/b_\perp}$ from (5):

$$
\begin{aligned}
\underline{x}_{b1\,p_1/a} &= \underline{x}_{b1\,p_1/a} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{b1\,p_1/a_\perp} = \alpha\,\underline{x}_{bp_1/b} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{bp_1/b_\perp} \\
\underline{x}_{b2\,q_2/a} &= \underline{x}_{b2\,q_2/a} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{b2\,q_2/a_\perp} = \alpha\,\underline{x}_{b\,q_2/b} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{b\,q_2/b_\perp}
\end{aligned}
\tag{18}
$$

where

α = Constant factor to account for the constancy in speed-of-light measurement law of Relativity theory, or for setting to unity for compatibility with classical Newtonian kinematics.

From Relativity theory, there is no preferred observation point (i.e., kinematic laws are the same for any observer), hence, the equivalent to (18) for observer b would be:

$$
\underline{x}_{a1\,p_1/b} = \alpha\,\underline{x}_{ap_1/a} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{ap_1/a_\perp} \qquad \underline{x}_{a2\,q_2/b} = \alpha\,\underline{x}_{a\,q_2/a} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{a\,q_2/a_\perp} \tag{19}
$$

Note that for α set to unity, both (18) and (19) become Newtonian theory equalities.

Substituting $\underline{x}_{p_1\,q_2/a}$ from (16) and $\underline{x}_{b1\,p_1/a}$, $\underline{x}_{b2\,q_2/a}$ from (18) into the (3) $\underline{x}_{p_1\,q_2/a}$ expression obtains

$$
\begin{aligned}
&\underline{x}_{p_1\,q_2/a} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{p_1\,q_2/a_\perp} \\
&= \alpha\,\underline{x}_{b\,q_2/b} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{b\,q_2/b_\perp} - \alpha\,\underline{x}_{bp_1/b} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon - \underline{x}_{b\,p_1/b_\perp} + \Delta\underline{x}_{ab,1\to2/a} \\
&= \alpha\left(\underline{x}_{b\,q_2/b} - \underline{x}_{bp_1/b}\right) \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{b\,q_2/b_\perp} - \underline{x}_{b\,p_1/b_\perp} + \Delta\underline{x}_{ab,1\to2/a} \\
&= \alpha\,\underline{x}_{p_1\,q_2/b} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{p_1\,q_2/b_\perp} + \Delta\underline{x}_{ab,1\to2/a}
\end{aligned}
\tag{20}
$$

Similarly, substituting $\underline{x}_{p_1\,q_2/b}$ from (16) and $\underline{x}_{a1\,p_1/b}$, $\underline{x}_{a2\,q_2/b}$ from (19) into the (3) $\underline{x}_{p_1\,q_2/b}$ expression finds

$$
\underline{x}_{p_1\,q_2/b} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{p_1\,q_2/b_\perp} = \alpha\,\underline{x}_{p_1\,q_2/a} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{p_1\,q_2/a_\perp} + \Delta\underline{x}_{ba,1\to2/b} \tag{21}
$$

Substituting $\Delta\underline{x}_{ab,1\to2/a}$ and $\Delta\underline{x}_{ba,1\to2/b}$ from (13) into (20) and (21), with rearrangement, then obtains a generalized version of the (15) Newtonian distance vector formulas that are compatible with either Newtonian or Relativity theory, depending on the value used for α:

$$
\begin{aligned}
\alpha\,\underline{x}_{p_1\,q_2/a} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{p_1\,q_2/a_\perp} &= \underline{x}_{p_1\,q_2/b} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{p_1\,q_2/b_\perp} + v_{ab}\,\Delta t_{b,1\to2}\,\underline{u}_\upsilon \\
\alpha\,\underline{x}_{p_1\,q_2/b} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{p_1\,q_2/b_\perp} &= \underline{x}_{p_1\,q_2/a} \cdot \underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{x}_{p_1\,q_2/a_\perp} - v_{ab}\,\Delta t_{a,1\to2}\,\underline{u}_\upsilon
\end{aligned}
\tag{22}
$$

The $\underline{u}_\upsilon$ components of (22) can be rearranged into

$$\underline{x}_{p_1 q_2/a} \cdot \underline{u}_\upsilon \underline{u}_\upsilon = \frac{1}{\alpha}\left(\underline{x}_{p_1 q_2/b} \cdot \underline{u}_\upsilon \underline{u}_\upsilon + v_{ab}\, \Delta t_{b,1\to 2}\, \underline{u}_\upsilon\right)$$
$$= \left(\underline{x}_{p_1 q_2/b} \cdot \underline{u}_\upsilon \underline{u}_\upsilon + v_{ab}\, \Delta t_{b,1\to 2}\, \underline{u}_\upsilon\right) + \left(\frac{1}{\alpha} - 1\right)\left(\underline{x}_{p_1 q_2/b} \cdot \underline{u}_\upsilon \underline{u}_\upsilon + v_{ab}\, \Delta t_{b,1\to 2}\, \underline{u}_\upsilon\right)$$
$$\underline{x}_{p_1 q_2/b} \cdot \underline{u}_\upsilon \underline{u}_\upsilon = \frac{1}{\alpha}\left(\underline{x}_{p_1 q_2/a} \cdot \underline{u}_\upsilon \underline{u}_\upsilon - v_{ab}\, \Delta t_{a,1\to 2}\, \underline{u}_\upsilon\right)$$
$$= \left(\underline{x}_{p_1 q_2/a} \cdot \underline{u}_\upsilon \underline{u}_\upsilon - v_{ab}\, \Delta t_{a,1\to 2}\, \underline{u}_\upsilon\right) + \left(\frac{1}{\alpha} - 1\right)\left(\underline{x}_{p_1 q_2/a} \cdot \underline{u}_\upsilon \underline{u}_\upsilon - v_{ab}\, \Delta t_{a,1\to 2}\, \underline{u}_\upsilon\right) \tag{23}$$

Substituting the $\underline{x}_{p_1 q_2/a} \cdot \underline{u}_\upsilon \underline{u}_\upsilon$ expression from (23) into (16) yields:

$$\underline{x}_{p_1 q_2/a} = \underline{x}_{p_1 q_2/b} \cdot \underline{u}_\upsilon \underline{u}_\upsilon + \underline{x}_{p_1 q_2/a_\perp} + v_{ab}\, \Delta t_{b,1\to 2}\, \underline{u}_\upsilon$$
$$+ \left(\frac{1}{\alpha} - 1\right)\left(\underline{x}_{p_1 q_2/b} \cdot \underline{u}_\upsilon \underline{u}_\upsilon + v_{ab}\, \Delta t_{b,1\to 2}\, \underline{u}_\upsilon\right) \tag{24}$$

Then, recognizing from the perpendicular components of (22) that $\underline{x}_{p_1 q_2/a_\perp} = \underline{x}_{p_1 q_2/b_\perp}$, and that from (16), $\underline{x}_{p_1 q_2/b} \cdot \underline{u}_\upsilon \underline{u}_\upsilon + \underline{x}_{p_1 q_2/b_\perp} = \underline{x}_{p_1 q_2/b}$, (24) becomes

$$\underline{x}_{p_1 q_2/a} = \underline{x}_{p_1 q_2/b} + v_{ab}\, \Delta t_{b,1\to 2}\, \underline{u}_\upsilon + \left(\frac{1}{\alpha} - 1\right)\left(\underline{x}_{p_1 q_2/b} \cdot \underline{u}_\upsilon \underline{u}_\upsilon + v_{ab}\, \Delta t_{b,1\to 2}\, \underline{u}_\upsilon\right) \tag{25}$$

Using the identical procedure for the $\underline{x}_{p_1 q_2/b} \cdot \underline{u}_\upsilon$ expression in (23), finds similarly for $\underline{x}_{p_1 q_2/b}$:

$$\underline{x}_{p_1 q_2/b} = \underline{x}_{p_1 q_2/a} - v_{ab}\, \Delta t_{a,1\to 2}\, \underline{u}_\upsilon + \left(\frac{1}{\alpha} - 1\right)\left(\underline{x}_{p_1 q_2/a} \cdot \underline{u}_\upsilon \underline{u}_\upsilon - v_{ab}\, \Delta t_{a,1\to 2}\, \underline{u}_\upsilon\right) \tag{26}$$

Eqs. (25) and (26) comprise a set of generalized distance vector conversion formulas (from observer *b* to *a* and from observer *a* to *b*) that are compatible with either Newtonian or Relativity theory. For a complete conversion set (as in the (15) Newtonian formulation) it remains to find generalized equations for converting $\Delta t_{b,1\to 2}$ to $\Delta t_{a,1\to 2}$ and $\Delta t_{a,1\to 2}$ to $\Delta t_{b,1\to 2}$.

Equations (25) and (26) can be inverted to find general solutions for the $\Delta t_{a,1\to 2}$ and $\Delta t_{b,1\to 2}$ time intervals. Taking the dot product of (26) with $\underline{u}_\upsilon$ obtains with rearrangement:

$$\underline{x}_{p_1 q_2/b} \cdot \underline{u}_\upsilon = \frac{1}{\alpha}\left(\underline{x}_{p_1 q_2/a} \cdot \underline{u}_\upsilon - v_{ab}\, \Delta t_{a,1\to 2}\right) \tag{27}$$

Substituting $\underline{x}_{p_1 q_2/b} \cdot \underline{u}_v$ from (27) into (25) (dotted into $\underline{u}_v$) and solving for t_b then gives:

$$\Delta t_{b,1\to2} = \frac{1}{\alpha}\left[\Delta t_{a,1\to2} - \left(1-\alpha^2\right) \underline{x}_{p_1 q_2/a} \cdot \underline{u}_v / v_{ab}\right] \tag{28}$$

Similarly, dotting (25) into $\underline{u}_v$ and substituting the $\underline{x}_{p_1 q_2/a} \cdot \underline{u}_v$ result into (26) (dotted into $\underline{u}_v$) solves for t_a :

$$\Delta t_{a,1\to2} = \frac{1}{\alpha}\left[\Delta t_{b,1\to2} + \left(1-\alpha^2\right) \underline{x}_{p_1 q_2/b} \cdot \underline{u}_v / v_{ab}\right] \tag{29}$$

Equations (25), (26), (28), and (29) summarize as follows

$$\begin{aligned}
\underline{x}_{p_1 q_2/a} &= \underline{x}_{p_1 q_2/b} + v_{ab}\, \Delta t_{b,1\to2}\, \underline{u}_v + \left(\frac{1}{\alpha}-1\right)\left(\underline{x}_{p_1 q_2/b} \cdot \underline{u}_v\, \underline{u}_v + v_{ab}\, \Delta t_{b,1\to2}\, \underline{u}_v\right) \\
\Delta t_{a,1\to2} &= \frac{1}{\alpha}\left[\Delta t_{b,1\to2} + \left(1-\alpha^2\right) \underline{x}_{p_1 q_2/b} \cdot \underline{u}_v / v_{ab}\right] \\
\underline{x}_{p_1 q_2/b} &= \underline{x}_{p_1 q_2/a} - v_{ab}\, \Delta t_{a,1\to2}\, \underline{u}_v + \left(\frac{1}{\alpha}-1\right)\left(\underline{x}_{p_1 q_2/a} \cdot \underline{u}_v\, \underline{u}_v - v_{ab}\, \Delta t_{a,1\to2}\, \underline{u}_v\right) \\
\Delta t_{b,1\to2} &= \frac{1}{\alpha}\left[\Delta t_{a,1\to2} - \left(1-\alpha^2\right) \underline{x}_{p_1 q_2/a} \cdot \underline{u}_v / v_{ab}\right]
\end{aligned} \tag{30}$$

Equations (30) constitute a generalized set of Point-to-Point kinematic conversion formulas that are compatible with either Newtonian or Relativity theory. The distinguishing characteristic between either is the value selected for the α constant. When α = 1, Eqs. (30) reduce to the classic (15) Newtonian form. For compatibility with Relativity theory, α must be set so that the speed-of-light constancy law is satisfied.

3.3 Setting Alpha For Relativity Compatibility

For compatibility with Relativity theory, α in (30) is used to account for experimental and theoretical findings that the speed of light (or any electro-magnetic wave speed in open space) is constant to any observer. Thus, consider what observers *a* and *b* would measure for the distance a photon of light would travel during the $\Delta t_{1\to2}$ time interval between *p* and *q* events. Defining the photon as point *r*, its spatial locations at time instants 1 and 2 as r_1 and r_2, and its travel direction parallel to $\underline{u}_v$ (i.e., parallel to $\underline{v}_{ab}$), the distance vector between r_1 and r_2 measured by the *a* and *b* observers would be:

$$\underline{x}_{r_1 r_2/a} = c\, \Delta t_{a,1\to2}\, \underline{u}_v \qquad \underline{x}_{r_1 r_2/b} = c\, \Delta t_{b,1\to2}\, \underline{u}_v \tag{31}$$

where

$\underline{x}_{r_1 r_2/a}$, $\underline{x}_{r_1 r_2/b}$ = Distance vector from photon point r location at time instant 1 to its location at time instant 2 as determined at observation points a and b.

c = Speed of light.

Identifying r_1 and r_2 as "events" in accordance with the Section 2.1 definition, and recognizing that general equations (30) apply for any two events occurring at time instants 1 and 2, (30) will remain valid with $\underline{x}_{r_1 r_2/a}$, $\underline{x}_{r_1 r_2/b}$ substituted for $\underline{x}_{p_1 q_2/a}$, $\underline{x}_{p_1 q_2/b}$. Thus, the distance vector equations in (30) will become

$$\begin{aligned}
\underline{x}_{r_1 r_2/a} &= \underline{x}_{r_1 r_2/b} + v_{ab}\, \Delta t_{b,1\to 2}\, \underline{u}_\upsilon + \left(\frac{1}{\alpha} - 1\right)\left(\underline{x}_{r_1 r_2/b} \cdot \underline{u}_\upsilon\, \underline{u}_\upsilon + v_{ab}\, \Delta t_{b,1\to 2}\, \underline{u}_\upsilon\right) \\
\underline{x}_{r_1 r_2/b} &= \underline{x}_{r_1 r_2/a} - v_{ab}\, \Delta t_{a,1\to 2}\, \underline{u}_\upsilon + \left(\frac{1}{\alpha} - 1\right)\left(\underline{x}_{r_1 r_2/a} \cdot \underline{u}_\upsilon\, \underline{u}_\upsilon - v_{ab}\, \Delta t_{a,1\to 2}\, \underline{u}_\upsilon\right)
\end{aligned} \qquad (32)$$

Substituting (31) in (32) and taking the dot product with $\underline{u}_\upsilon$ then obtains

$$\begin{aligned}
c\, \Delta t_{a,1\to 2} &= \frac{1}{\alpha}\left(c\, \Delta t_{b,1\to 2} + v_{ab}\, \Delta t_{b,1\to 2}\right) = \frac{1}{\alpha}\left(c + v_{ab}\right) \Delta t_{b,1\to 2} \\
c\, \Delta t_{b,1\to 2} &= \frac{1}{\alpha}\left(c\, \Delta t_{a,1\to 2} + v_{ab}\, \Delta t_{a,1\to 2}\right) = \frac{1}{\alpha}\left(c - v_{ab}\right) \Delta t_{a,1\to 2}
\end{aligned} \qquad (33)$$

or

$$\begin{aligned}
\alpha\, \Delta t_{b,1\to 2} &= \left(1 - v_{ab} / c\right) \Delta t_{a,1\to 2} \qquad \therefore \frac{\Delta t_{b,1\to 2}}{\Delta t_{a,1\to 2}} = \frac{1 - v_{ab} / c}{\alpha} \\
\alpha\, \Delta t_{a,1\to 2} &= \left(1 + v_{ab} / c\right) \Delta t_{b,1\to 2} \qquad \therefore \frac{\Delta t_{b,1\to 2}}{\Delta t_{a,1\to 2}} = \frac{\alpha}{1 + v_{ab} / c}
\end{aligned} \qquad (34)$$

Equating the $\dfrac{\Delta t_{b,1\to 2}}{\Delta t_{a,1\to 2}}$ expressions in (34) solves for α^2:

$$\frac{1 - v_{ab} / c}{\alpha} = \frac{\alpha}{1 + v_{ab} / c} \qquad \therefore \quad \alpha^2 = \left(1 - v_{ab} / c\right)\left(1 + v_{ab} / c\right) = 1 - v_{ab}^2 / c^2 \qquad (35)$$

Hence, from (35),

$$1 - \alpha^2 = v_{ab}^2 / c^2 \qquad (36)$$

and Relativity coefficient α becomes the well-known form:

$$\alpha = \sqrt{1 - v_{ab}^2 / c^2} \qquad (37)$$

3.4 Point-To-Point Lorentz Conversion Formulas

With α now in hand from (37), substitution in (30) with (36) for $1-\alpha^2$ obtains the general Point-to-Point conversion formulas:

$$
\begin{aligned}
&\underline{x}_{p_1 q_2/a} = \underline{x}_{p_1 q_2/b} + v_{ab}\,\Delta t_{b,1\to 2}\,\underline{u}_\upsilon + \left(\frac{1}{\sqrt{1-v_{ab}^2/c^2}}-1\right)\left(\underline{x}_{p_1 q_2/b}\cdot\underline{u}_\upsilon + v_{ab}\,\Delta t_{b,1\to 2}\right)\underline{u}_\upsilon \\
&\qquad \Delta t_{a,1\to 2} = \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(\Delta t_{b,1\to 2} + \underline{x}_{p_1 q_2/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right) \\
&\underline{x}_{p_1 q_2/b} = \underline{x}_{p_1 q_2/a} - v_{ab}\,\Delta t_{a,1\to 2}\,\underline{u}_\upsilon + \left(\frac{1}{\sqrt{1-v_{ab}^2/c^2}}-1\right)\left(\underline{x}_{p_1 q_2/a}\cdot\underline{u}_\upsilon - v_{ab}\,\Delta t_{a,1\to 2}\right)\underline{u}_\upsilon \\
&\qquad \Delta t_{b,1\to 2} = \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(\Delta t_{a,1\to 2} - \underline{x}_{p_1 q_2/a}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)
\end{aligned}
\tag{38}
$$

Equations (38) can be converted to a more traditional form by first substituting $\underline{u}_\upsilon = \underline{v}_{ab/a}/v_{ab}$ from (9) as follows:

$$
\begin{aligned}
&\left(\underline{x}_{p_1 q_2/b}\cdot\underline{u}_\upsilon + v_{ab}\,\Delta t_{b,1\to 2}\right)\underline{u}_\upsilon = \underline{x}_{p_1 q_2/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon + \underline{v}_{ab/a}\,\Delta t_{b,1\to 2} \\
&= \underline{x}_{p_1 q_2/b}\cdot\underline{v}_{ab/a}\,\underline{v}_{ab/a}/v_{ab}^2 + \underline{v}_{ab/a}\,\Delta t_{b,1\to 2} = \left(\underline{x}_{p_1 q_2/b}\cdot\underline{v}_{ab/a}/v_{ab}^2 + \Delta t_{b,1\to 2}\right)\underline{v}_{ab/a} \\
&= \left(\underline{x}_{p_1 q_2/b}\cdot\underline{v}_{ab/a} + v_{ab}^2\,\Delta t_{b,1\to 2}\right)\underline{v}_{ab/a}/v_{ab}^2
\end{aligned}
$$

$$
v_{ab}^2 = \underline{v}_{ab/a}\cdot\underline{v}_{ab/a} \qquad v_{ab}^2\,\Delta t_{b,1\to 2} = \left(\underline{v}_{ab/a}\,\Delta t_{b,1\to 2}\right)\cdot\underline{v}_{ab/a} \tag{39}
$$

$$
\left(\underline{x}_{p_1 q_2/b}\cdot\underline{v}_{ab/a} + v_{ab}^2\,\Delta t_{b,1\to 2}\right)\underline{v}_{ab/a}/v_{ab}^2 = \left(\underline{x}_{p_1 q_2/b} + \underline{v}_{ab/a}\,\Delta t_{b,1\to 2}\right)\cdot\underline{v}_{ab/a}\,\underline{v}_{ab/a}/v_{ab}^2
$$

$$\therefore$$

$$
\begin{aligned}
&\left(\underline{x}_{p_1 q_2/b}\cdot\underline{u}_\upsilon + v_{ab}\,\Delta t_{b,1\to 2}\right)\underline{u}_\upsilon = \left(\underline{x}_{p_1 q_2/b} + \underline{v}_{ab/a}\,\Delta t_{b,1\to 2}\right)\cdot\underline{v}_{ab/a}\,\underline{v}_{ab/a}/v_{ab}^2 \\
&\left(\underline{x}_{p_1 q_2/a}\cdot\underline{u}_\upsilon - v_{ab}\,\Delta t_{a,1\to 2}\right)\underline{u}_\upsilon = \left(\underline{x}_{p_1 q_2/a} - \underline{v}_{ab/a}\,\Delta t_{a,1\to 2}\right)\cdot\underline{v}_{ab/a}\,\underline{v}_{ab/a}/v_{ab}^2
\end{aligned}
$$

Using the (39) result, $\underline{u}_\upsilon = \underline{v}_{ab/a}/v_{ab}$ from (9), and recognizing from (12) that $\underline{v}_{ab/a} = \underline{v}_{ab/b}$, Eqs. (38) then become

$$\begin{aligned}
\underline{x}_{p_1 q_2/a} &= \underline{x}_{p_1 q_2/b} + \underline{v}_{ab/b}\, \Delta t_{b,1\to 2} \\
&\quad + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(\underline{x}_{p_1 q_2/b} + \underline{v}_{ab/b}\, \Delta t_{b,1\to 2} \right) \cdot \underline{v}_{ab/b}\, \underline{v}_{ab/b} / v_{ab}^2 \\
\Delta t_{a,1\to 2} &= \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(\Delta t_{b,1\to 2} + \underline{x}_{p_1 q_2/b} \cdot \underline{v}_{ab/b} / c^2 \right) \\
\underline{x}_{p_1 q_2/b} &= \underline{x}_{p_1 q_2/a} - \underline{v}_{ab/a}\, \Delta t_{a,1\to 2} \\
&\quad + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(\underline{x}_{p_1 q_2/a} - \underline{v}_{ab/a}\, \Delta t_{a,1\to 2} \right) \cdot \underline{v}_{ab/a}\, \underline{v}_{ab/a} / v_{ab}^2 \\
\Delta t_{b,1\to 2} &= \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(\Delta t_{a,1\to 2} - \underline{x}_{p_1 q_2/a} \cdot \underline{v}_{ab/a} / c^2 \right)
\end{aligned} \tag{40}$$

Equations (40) are the Point-to-Point conversion equivalent of the general Lorentz transformation operations in traditional Relativity theory [9 pp. 30].

3.4.1 Point-to-Point Geometrical Interpretation of Lorentz Conversion

For almost all applications, approximating $\sqrt{1 - v_{ab}^2 / c^2}$ by unity (i.e., α in general equations (30) set to one as in Newtonian kinematics), creates negligible error. For example, if v_{ab} = 1.E3 meters per second (and as usual, c = 3.E8 meters per second), $\alpha = 1 - 0.5\, v_{ab}^2 / c^2 + \cdots$ $= 1 - 5.6\text{E-}12 + \cdots \approx 1$. For situations when v_{ab} is unusually large (e.g., particle motion generated by a cyclotron), or when very small time intervals are important to system accuracy (e.g., Global Positioning System - GPS pseudo-range measurements), approximating α by unity is not acceptable, and the full $\sqrt{1 - v_{ab}^2 / c^2}$ forms must be used.

To illustrate the effect of $\sqrt{1 - v_{ab}^2 / c^2}$ deviating significantly from unity, consider a case when $v_{ab} = 0.866\, c$ (corresponding to $\sqrt{1 - v_{ab}^2 / c^2} = 0.5$ and $v_{ab}^2 / c^2 = 0.75$), and when observer b sees the p and q events at time instants 1 and 2 occurring simultaneously (i.e., $\Delta t_{b,1\to 2} = 0$). Under this condition, (38) shows that $\underline{x}_{p_1 q_2/a} = \underline{x}_{p_1 q_2/b} + \underline{x}_{p_1 q_2/b} \cdot \underline{u}_v\, \underline{u}_v$. Fig. 5a illustrates the vector construction of $\underline{x}_{p_1 q_2/a}$ from $\underline{x}_{p_1 q_2/b}$ using this formula.

The equivalent construction of $\underline{x}_{p_1 q_2/b}$ from $\underline{x}_{p_1 q_2/a}$ under the previous conditions, derives from the last two equations in (38). The $\Delta t_{b,1\to 2}$ equation in (38) shows that for $v_{ab}^2 / c^2 = 0.75$ and $\Delta t_{b,1\to 2} = 0$, the time interval between the p and q events observed on the point a clock will be $\Delta t_{a,1\to 2} = 0.75\, \underline{x}_{p_1 q_2/a} \cdot \underline{u}_{\upsilon} / v_{ab}$. Substituting this with $\sqrt{1 - v_{ab}^2 / c^2} = 0.5$ into the (38) $\underline{x}_{p_1 q_2/b}$ equation finds that $\underline{x}_{p_1 q_2/b} = \underline{x}_{p_1 q_2/a} - 0.75\, \underline{x}_{p_1 q_2/a} \cdot \underline{u}_{\upsilon}\, \underline{u}_{\upsilon}$ $+\left(\underline{x}_{p_1 q_2/a} \cdot \underline{u}_{\upsilon} - 0.75\, \underline{x}_{p_1 q_2/a} \cdot \underline{u}_{\upsilon}\right) \underline{u}_{\upsilon} = \underline{x}_{p_1 q_2/a} - 0.5\, \underline{x}_{p_1 q_2/a} \cdot \underline{u}_{\upsilon}\, \underline{u}_{\upsilon}$. Fig. 5b illustrates how this result can be used to reconstruct $\underline{x}_{p_1 q_2/b}$ from the $\underline{x}_{p_1 q_2/a}$ value in Fig. 5a, thereby regenerating the original Fig. 5a $\underline{x}_{p_1 q_2/b}$ vector.

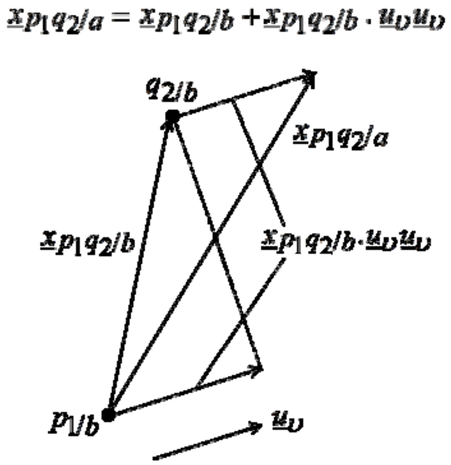

Fig. 5a - Constructing $\underline{x}_{p_1 q_2/a}$
From $\underline{x}_{p_1 q_2/b}$ When $\Delta t_{b,1\to 2} = 0$

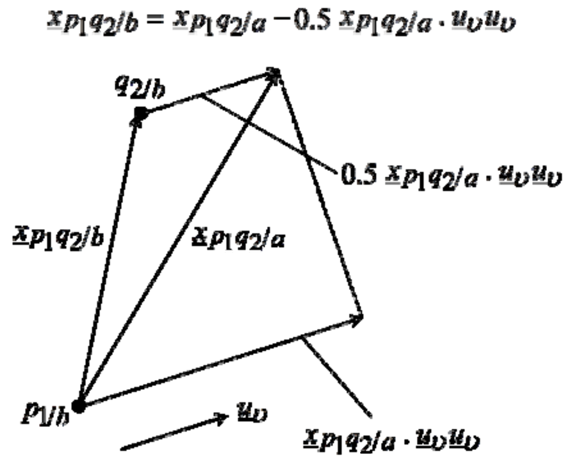

Fig. 5b - Constructing $\underline{x}_{p_1 q_2/b}$
From $\underline{x}_{p_1 q_2/a}$ When $\Delta t_{b,1\to 2} = 0$

A similar analysis/construction can be performed for a case when the p and q events at time instants 1 and 2 occur simultaneously at point a (i.e., $\Delta t_{a,1\to 2} = 0$), leading to similar results (left as an exercise for the reader).

3.4.2 Simplified Point-to-Point Lorentz Conversion Notation

Now that the meaning of time and distance parameters in (40) have been clearly established, a simplified notation can be introduced that deletes the time instant 1 and 2 notation, treating all parameters as general variables:

$$\underline{x}_{pq/a} = \underline{x}_{pq/b} + \underline{v}_{ab/b}\, \Delta t_{pq/b} + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(\underline{x}_{pq/b} + \underline{v}_{ab/b}\, \Delta t_{pq/b} \right) \cdot \underline{v}_{ab/b}\, \underline{v}_{ab/b} / v_{ab}^2$$

$$\Delta t_{pq/a} = \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(\Delta t_{pq/b} + \underline{x}_{pq/b} \cdot \underline{v}_{ab/b} / c^2 \right) \tag{41}$$

Continued

(41) Concluded

$$\underline{x}_{pq/b} = \underline{x}_{pq/a} - \underline{v}_{ab/a}\,\Delta t_{pq/a} + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(\underline{x}_{pq/a} - \underline{v}_{ab/a}\,\Delta t_{pq/a} \right) \cdot \underline{v}_{ab/a}\,\underline{v}_{ab/a} / v_{ab}^2$$

$$\Delta t_{pq/b} = \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(\Delta t_{pq/a} - \underline{x}_{pq/a} \cdot \underline{v}_{ab/a} / c^2 \right)$$

where

$\underline{x}_{pq/a}$, $\underline{x}_{pq/b}$ = Distance vector from event point p to event point q based on observations of the events at points a and b.

$\Delta t_{pq/a}$, $\Delta t_{pq/b}$ = Time interval between the p and q events based on time measurements for the events at points a and b.

3.4.3 Point-To-Point Lorentz Conversion Vector Components

Most past works in Relativity theory have expressed derivations and results in terms of component projections on coordinate frame axes. Equivalent results are generated with the Point-to-Point approach by first expressing Poin-to-Pointt conversion formulas (41) in equivalent matrix format as described in Appendix B. For example, consider two inertially non-rotating coordinate frames, one at angular orientation relative to the other. Define the first as coordinate frame A in which the X axis is aligned with the point b to point a velocity vector $\underline{v}_{ab/a}$, the second as coordinate frame B at some general angular orientation relative to frame A. Appendix B shows how the angular orientations between the two frames can be represented as a direction cosine matrix in which rows and columns represent unit vectors along the coordinate frame axes. Based on the A and B frame definitions, we then define the components of $\underline{x}_{pq/a}$, $\underline{x}_{pq/b}$, and $\underline{v}_{ab/a}$ as elements of column matrices:

$$\underline{x}_{pq/a}^{\mathrm{A}} \equiv \begin{bmatrix} X_{\mathrm{A}/a} \\ Y_{\mathrm{A}/a} \\ Z_{\mathrm{A}/a} \end{bmatrix} \quad \underline{x}_{pq/b}^{\mathrm{A}} \equiv \begin{bmatrix} X_{\mathrm{A}/b} \\ Y_{\mathrm{A}/b} \\ Z_{\mathrm{A}/b} \end{bmatrix} \quad \underline{x}_{pq/a}^{\mathrm{B}} \equiv \begin{bmatrix} X_{\mathrm{B}/a} \\ Y_{\mathrm{B}/a} \\ Z_{\mathrm{B}/a} \end{bmatrix} \quad \underline{x}_{pq/b}^{\mathrm{B}} \equiv \begin{bmatrix} X_{\mathrm{B}/b} \\ Y_{\mathrm{B}/b} \\ Z_{\mathrm{B}/b} \end{bmatrix}$$

$$\underline{v}_{ab/a}^{\mathrm{A}} = \underline{v}_{ab/b}^{\mathrm{A}} \equiv \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} \quad \underline{v}_{ab/a}^{\mathrm{B}} = \underline{v}_{ab/b}^{\mathrm{B}} \equiv \begin{bmatrix} V_{\mathrm{XB}} \\ V_{\mathrm{YB}} \\ V_{\mathrm{ZB}} \end{bmatrix} \quad V \equiv \sqrt{\underline{v}_{ab/a}^{\mathrm{A}} \cdot \underline{v}_{ab/a}^{\mathrm{A}}} = \sqrt{\underline{v}_{ab/a}^{\mathrm{B}} \cdot \underline{v}_{ab/a}^{\mathrm{B}}} \qquad (42)$$

$$V^2 = V_{XB}^2 + V_{YB}^2 + V_{ZB}^2$$

where

$(\ )^{\mathrm{A}}$, $(\ )^{\mathrm{B}}$ = Column matrices with elements equal to the projections of vector $(\ )$ on coordinate frame A and B axes. Note - Vector $(\ )$ projections on coordinate frame A or B axes are defined as the dot product of ( ) with the mutually orthogonal unit vectors that define coordinate frames A and B.

V = Magnitude of $\underline{v}_{ab/a}$ (and of $\underline{v}_{ab/b}$ since from (12), they are equal).

Equations (41) are then converted to the equivalent matrix form by replacing all vectors with their equivalent column matrix forms (as in (42) by adding coordinate frame designation superscripts A or B for conversion to A or B frame coordinates), replacing all ( ) column matrix dot-product operations ( ). with the matrix transpose operator equivalent $(\ )^T$, and replacing all cross-product operations $(\ )\times$ with the square matrix operator equivalent $\left[(\ )\times\right]$ defined for a general vector $\underline{W}$ in the A frame as $\underline{W}^{\mathrm{A}} = \begin{bmatrix} W_{\mathrm{XA}} \\ W_{\mathrm{YA}} \\ W_{\mathrm{ZA}} \end{bmatrix}$ $\quad \left[\underline{W}^{\mathrm{A}}\times\right] = \begin{bmatrix} 0 & -W_{\mathrm{ZA}} & W_{\mathrm{YA}} \\ W_{\mathrm{ZA}} & 0 & -W_{\mathrm{XA}} \\ -W_{\mathrm{YA}} & W_{\mathrm{XA}} & 0 \end{bmatrix}$ and similarly for $\underline{W}^{\mathrm{B}}$. It is easily shown by standard matrix algebra that for an arbitrary vector $\underline{U}$, the column matrix form of cross-product vector $\underline{W}\times\underline{U}$ in the A frame, i.e., $\left(\underline{W}\times\underline{U}\right)^{\mathrm{A}}$, equals $\left[\underline{W}^{\mathrm{A}}\times\right]\underline{U}^{\mathrm{A}}$ (and similarly for the B frame). The individual components of (41) projected on coordinate frame A or B axes are then obtained by substituting (42) for column matrices in the converted (41) matrix equations, carrying out the indicated matrix operations, and identifying the rows of the result as the equations for individual components. The result for (41) projections on A frame axes (including $\Delta t_{pq/a}$ and $\Delta t_{pq/b}$) is

$$
\begin{aligned}
X_{\mathrm{A}/a} &= \frac{1}{\sqrt{1-V^2/c^2}}\left(X_{\mathrm{A}/b} + V\,\Delta t_{pq/b}\right) \qquad Y_{\mathrm{A}/a} = Y_{\mathrm{A}/b} \qquad Z_{\mathrm{A}/a} = Z_{\mathrm{A}/b} \\
\Delta t_{pq/a} &= \frac{1}{\sqrt{1-V^2/c^2}}\left(\Delta t_{pq/b} + V\,X_{\mathrm{A}/b}/c^2\right) \\
X_{\mathrm{A}/b} &= \frac{1}{\sqrt{1-V^2/c^2}}\left(X_{\mathrm{A}/a} - V\,\Delta t_{pq/a}\right) \qquad Y_{\mathrm{A}/b} = Y_{\mathrm{A}/a} \qquad Z_{\mathrm{A}/b} = Z_{\mathrm{A}/a} \\
\Delta t_{pq/b} &= \frac{1}{\sqrt{1-V^2/c^2}}\left(\Delta t_{pq/a} - V\,X_{\mathrm{A}/a}/c^2\right)
\end{aligned}
\tag{43}
$$

The (43) form is what is known in Relativity theory as the “standard configuration” of the Lorentz Transformation [2 pp. 37, 7 Eqs. (12-7a) & (12-7b), 8 Eq. (70a) & (70b), 9 Eqs. (10.41) - (10.44)]. In traditional Relativity theory, the *a* and *b* observer distance vector components and time intervals in (43) would correspond to what would be measured in two coordinate frames that are translating at velocity *V* relative to each other in the X coordinate axis direction.

Following the same procedure used to obtain the (41) frame A components, projections of (41) on B frame coordinate axes finds

$$X_{\mathrm{B}/a} = X_{\mathrm{B}/b} + V_{\mathrm{XB}}\,\Delta t_{pq/b} + \left(\frac{1}{\sqrt{1-V^2/c^2}} - 1\right)\frac{V_{\mathrm{XB}}}{V^2}\begin{pmatrix} V_{\mathrm{XB}}\left(X_{\mathrm{B}/b} + V_{\mathrm{XB}}\,\Delta t_{pq/b}\right) \\ +V_{\mathrm{YB}}\left(X_{\mathrm{Y}/b} + V_{\mathrm{YB}}\,\Delta t_{pq/b}\right) \\ +V_{\mathrm{ZB}}\left(X_{\mathrm{Z}/b} + V_{\mathrm{ZB}}\,\Delta t_{pq/b}\right) \end{pmatrix}$$

$$Y_{\mathrm{B}/a} = Y_{\mathrm{B}/b} + V_{\mathrm{YB}}\,\Delta t_{pq/b} + \left(\frac{1}{\sqrt{1-V^2/c^2}} - 1\right)\frac{V_{\mathrm{YB}}}{V^2}\begin{pmatrix} V_{\mathrm{XB}}\left(X_{\mathrm{B}/b} + V_{\mathrm{XB}}\,\Delta t_{pq/b}\right) \\ +V_{\mathrm{YB}}\left(X_{\mathrm{Y}/b} + V_{\mathrm{YB}}\,\Delta t_{pq/b}\right) \\ +V_{\mathrm{ZB}}\left(X_{\mathrm{Z}/b} + V_{\mathrm{ZB}}\,\Delta t_{pq/b}\right) \end{pmatrix} \qquad (44)$$

$$Z_{\mathrm{B}/a} = Z_{\mathrm{B}/b} + V_{\mathrm{ZB}}\,\Delta t_{pq/b} + \left(\frac{1}{\sqrt{1-V^2/c^2}} - 1\right)\frac{V_{\mathrm{ZB}}}{V^2}\begin{pmatrix} V_{\mathrm{XB}}\left(X_{\mathrm{B}/b} + V_{\mathrm{XB}}\,\Delta t_{pq/b}\right) \\ +V_{\mathrm{YB}}\left(X_{\mathrm{Y}/b} + V_{\mathrm{YB}}\,\Delta t_{pq/b}\right) \\ +V_{\mathrm{ZB}}\left(X_{\mathrm{Z}/b} + V_{\mathrm{ZB}}\,\Delta t_{pq/b}\right) \end{pmatrix}$$

$$\Delta t_{pq/a} = \frac{1}{\sqrt{1-V^2/c^2/c^2}}\left(\Delta t_{pq/b} + \frac{1}{c^2}\left(V_{\mathrm{XB}}\,X_{\mathrm{B}/b} + V_{\mathrm{YB}}\,Y_{\mathrm{B}/b} + V_{\mathrm{ZB}}\,Z_{\mathrm{B}/b}\right)\right)$$

Equations (44) are the general version of the Lorentz Transformation [7 Eqs. (12-5a), 9 Eqs. (10.32) - (10.33) & (10-36) - (10.37)]. In traditional Relativity theory, the *a* and *b* observer distance vector components and time intervals in (44) would correspond to what would be measured in two coordinate frames translating relative to each other at relative velocity magnitude *V* and components V_{XB}, V_{YB}, V_{ZB}. The same procedure leading to (44) would be used to obtain B frame $\underline{x}_{pq/b}$ components and $\Delta t_{pq/b}$ in terms of A frame $\underline{x}_{pq/a}$ components and $\Delta t_{pq/a}$.

Equations (44) can also be derived by transforming (43) from the A frame to the B frame using an A to B frame direction cosine matrix (left as an exercise for the reader - see Appendix B). It is interesting to note that [9 Sect. 10] used the direction cosine transformation method to derive (44), and that the (41) general vector form was then deduced from (44). In comparison, this article used a direct vector design approach to derive general vector equations (41) with the (43) and (44) vector component equations then obtained by projection of (41) on coordinate frame A and B axes.

4.0 THREE-POINT AND TWO-POINT FORMULATIONS

The analytical developments thus far have been based on a four-point approach; two observation points and two distinct points where events occur. This section analyzes cases where events occur at the same point but at different times; a three-point approach where the event point is remote from the observation points, and a two-point formulation in which events occur at one of the observation points.

4.1 <u>Three-Point Approach</u>

Consider a case in which event point q at time instant 2 is defined to be point p that has moved from its time instant 1 location to a new location at time instant 2. Using this concept, Point-to-Point Lorentz conversion formulas (40) become:

$$
\begin{aligned}
\Delta\underline{x}_{p,1\to 2/a} &= \Delta\underline{x}_{p,1\to 2/b} + \underline{v}_{ab/b}\,\Delta t_{b,1\to 2} \\
&\quad + \left(\frac{1}{\sqrt{1-v_{ab}^2/c^2}} - 1\right)\left(\Delta\underline{x}_{p,1\to 2/b} + \underline{v}_{ab/b}\,\Delta t_{b,1\to 2}\right)\cdot\underline{v}_{ab/b}\,\underline{v}_{ab/b}/v_{ab}^2 \\
\Delta t_{a,1\to 2} &= \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(\Delta t_{b,1\to 2} + \Delta\underline{x}_{p,1\to 2/b}\cdot\underline{v}_{ab/b}/c^2\right) \\
\Delta\underline{x}_{p,1\to 2/b} &= \Delta\underline{x}_{p,1\to 2/a} - \underline{v}_{ab/a}\,\Delta t_{a,1\to 2} \\
&\quad + \left(\frac{1}{\sqrt{1-v_{ab}^2/c^2}} - 1\right)\left(\Delta\underline{x}_{p,1\to 2/a} - \underline{v}_{ab/a}\,\Delta t_{a,1\to 2}\right)\cdot\underline{v}_{ab/a}\,\underline{v}_{ab/a}/v_{ab}^2 \\
\Delta t_{b,1\to 2} &= \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(\Delta t_{a,1\to 2} - \Delta\underline{x}_{p,1\to 2/a}\cdot\underline{v}_{ab/a}/c^2\right)
\end{aligned}
\tag{45}
$$

where

$\Delta\underline{x}_{p,1\to 2/a}$, $\Delta\underline{x}_{p,1\to 2/b}$ = Change in position location of point p from time instant 1 to time instant 2 as observed at points a and b.

$\Delta t_{a,1\to 2}$, $\Delta t_{b,1\to 2}$ = Time interval time for the $\Delta\underline{x}_{p,1\to 2/a}$, $\Delta\underline{x}_{p,1\to 2/b}$ point p position changes as measured on the point a and b clocks.

Similar to (41), the point 1 and 2 distinction notation in (45) can then be eliminated to obtain the simplified notation version:

$$
\begin{aligned}
\Delta\underline{x}_{p/a} &= \Delta\underline{x}_{p/b} + \underline{v}_{ab/b}\,\Delta t_b + \left(\frac{1}{\sqrt{1-v_{ab}^2/c^2}} - 1\right)\left(\Delta\underline{x}_{p/b} + \underline{v}_{ab/b}\,\Delta t_b\right)\cdot\underline{v}_{ab/b}\,\underline{v}_{ab/b}/v_{ab}^2 \\
\Delta t_a &= \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(\Delta t_b + \Delta\underline{x}_{p/b}\cdot\underline{v}_{ab/b}/c^2\right)
\end{aligned}
\tag{46}
$$

Continued

(46) Concluded

$$\Delta \underline{x}_{p/b} = \Delta \underline{x}_{p/a} - \underline{v}_{ab/a}\,\Delta t_a + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1\right)\left(\Delta \underline{x}_{p/a} - \underline{v}_{ab/a}\,\Delta t_a\right) \cdot \underline{v}_{ab/a}\,\underline{v}_{ab/a} / v_{ab}^2$$

$$\Delta t_b = \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}}\left(\Delta t_a - \Delta \underline{x}_{p/a} \cdot \underline{v}_{ab/a} / c^2\right)$$

where

Δt_a, Δt_b = Time interval measured on the point *a* and *b* clocks between point *a* and *b* observed point *p* position changes.

$\Delta \underline{x}_{p/a}$, $\Delta \underline{x}_{p/b}$ = Change in the vector position of point *p* over the Δt_a, Δt_b time intervals based on point *p* position observations made at points *a* and *b*.

4.2 Two-Point Approach

This section analyzes a two-point version of the Point-to-Point Lorentz conversion formulas in which the first event at point *p* occurs at observation point *b* at time instant 1, and the second event at point *q* occurs at observation point *b* at time instant 2, i.e., the events occur at point *b* that moves from its location at time instant 1 to its location at time instant 2. Then $\underline{x}_{p_1\, q_2/b}$ in (40) becomes $\underline{x}_{b_1\, b_2/b}$ which equals zero at point *b*, hence, Point-to-Point Lorentz conversion formulas (45) become

$$\underline{x}_{b_1\, b_2/a} = \underline{v}_{ab/b}\,\Delta t_{b,b_1 \to b_2} + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1\right)\left(\underline{v}_{ab/b}\,\Delta t_{b,b_1 \to b_2}\right) \cdot \underline{v}_{ab/b}\,\underline{v}_{ab/b} / v_{ab}^2$$

$$= \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}}\,\underline{v}_{ab/b}\,\Delta t_{b,b_1 \to b_2}$$

$$\Delta t_{a,b_1 \to b_2} = \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}}\,\Delta t_{b,b_1 \to b_2} \quad \Rightarrow \quad \Delta t_{b,b_1 \to b_2} = \sqrt{1 - v_{ab}^2 / c^2}\;\Delta t_{a,b_1 \to b_2} \qquad (47)$$

$$\underline{x}_{b_1\, b_2/b} = 0$$

$$\Delta t_{b,b_1 \to b_2} = \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}}\left(\Delta t_{a,b_1 \to b_2} - \underline{x}_{b_1\, b_2/a} \cdot \underline{v}_{ab/a} / c^2\right)$$

where

$\Delta t_{a,b1\to b2}$, $\Delta t_{b,b1\to b2}$ = Time interval between the point *b* event at time instant 1 and the point *b* event at time instant 2, as measured on the point *a* and *b* clocks.

$\underline{x}_{b1\,b2/a}$, $\underline{x}_{b1\,b2/b}$ = Distance vector from the point *b* event at time instant 1 to the point *b* event at time instant 2, as observed at points *a* and *b*.

Substituting the $\Delta t_{b,b1\to b2}$ result from the second (47) equation into the (47) $\underline{x}_{b1\,b2/a}$ equation with $\underline{v}_{ab/b} = \underline{v}_{ab/a}$ from (12) obtains:

$$\underline{x}_{b1\,b2/a} = \underline{v}_{ab/a}\,\Delta t_{a,b1\to b2} \qquad (48)$$

For the assumed constant $\underline{v}_{ab/a}$, (48) is the integral of $\underline{v}_{ab/a}$ in (8) over the $\Delta t_{a,b1\to b2}$ time interval. The $\Delta t_{a,b1\to b2}$ expression from (47) and its substitution in (48) gives

$$\underline{x}_{b1\,b2/a} = \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\,\underline{v}_{ab/a}\,\Delta t_{b,b1\to b2} \qquad \Delta t_{a,b1\to b2} = \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\,\Delta t_{b,b1\to b2} \qquad (49)$$

Similar to (48), the integral of $\underline{v}_{ba/b}$ in (8) (with $\underline{v}_{ba/b} = -\,\underline{v}_{ab/a}$) over the $\Delta t_{b,b1\to b2}$ time period shows that, as in (13):

$$\Delta\underline{x}_{ba,b1\to b2/b} = -\,\underline{v}_{ab}\,\Delta t_{b,b1\to b2} \qquad (50)$$

where

$\Delta\underline{x}_{ba,b1\to b2/b}$ = Change in the distance vector from point *b* to *a* during the time interval between point *b* events at time instants 1 and 2, as determined at point *b* - Similar to $\Delta\underline{x}_{ba,1\to 2/b}$ in Eq. (4).

The $\Delta t_{b,b1\to b2}$ expression from (47) and its substitution in (50) shows that:

$$\Delta\underline{x}_{ba,b1\to b2/b} = -\sqrt{1-v_{ab}^2/c^2}\,\underline{v}_{ab/a}\,\Delta t_{a,b1\to b2} \qquad \Delta t_{b,b1\to b2} = \sqrt{1-v_{ab}^2/c^2}\,\Delta t_{a,b1\to b2} \qquad (51)$$

Equations (49) and (51) describe observations at points *a* and *b* of the other's distance and time interval measurements, both based on time instant 1 and 2 events occurring at observation point *b*. An analysis similar to that leading to (49) and (51) can also be performed for a two-point case in which the events occur at point *a*. For clarity, we define the point *a* event time instants to be $1'$ followed by $2'$, but independent of sequential time instants 1 and 2 (which in this section designate events occurring at point *b*). (Note that time instants $1'$ or $2'$ may occur before or after 1 or 2). With the $1', 2'$ designation for point *a* events, $\underline{x}_{p_1\,q_2/a}$ in (40) becomes $\underline{x}_{a1'\,a2'/a}$ which equals zero). Assuming the same constant velocity $\underline{v}_{ab/a}$ between points *a* and *b* for events occurring at either *a* and/or *b*, the equivalent to the (48), (49), and (51) results would then be

$$\underline{x}_{a1'\,a2'/b} = -\,\underline{v}_{ab/a}\,\Delta t_{b,a1'\to a2'} \tag{52}$$

$$\underline{x}_{a1'\,a2'/b} = -\frac{1}{\sqrt{1-v_{ab}^2/c^2}}\,\underline{v}_{ab/a}\,\Delta t_{a,a1'\to a2'} \qquad \Delta t_{b,a1'\to a2'} = \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\,\Delta t_{a,a1'\to a2'} \tag{53}$$

$$\Delta\underline{x}_{ab,a1'\to a2'/a} = \sqrt{1-v_{ab}^2/c^2}\;\underline{v}_{ab/a}\,\Delta t_{b,a1'\to a2'} \qquad \Delta t_{a,a1'\to a2'} = \sqrt{1-v_{ab}^2/c^2}\;\Delta t_{b,a1'\to a2'} \tag{54}$$

where

$\Delta t_{a,a1'\to a2'}$, $\Delta t_{b,a1'\to a2'}$ = Time interval between the first point *a* event at time instant $1'$ and the second point *a* event at time instant $2'$, as measured on the point *a* and *b* clocks.

$\underline{x}_{a1'\,a2'/b}$ = Distance vector from the point *a* event at time instant $1'$ to the point *a* event at time instant $2'$, as observed at point *b*.

$\Delta\underline{x}_{ab,a1'\to a2'/a}$ = Change in the distance vector from point *a* to *b* during the time interval between point *a* events at time instants $1'$ and $2'$, as determined at point *a* - Similar to $\Delta\underline{x}_{ab,1\to 2/a}$ in Eq. (4).

Eqs. (49) and (53) are equivalent to what has been derived using traditional Relativity theory, e.g. [7 pp. 517], demonstrating the difference in observations when conditions defining the events are similar, but different.

5.0 POINT-TO-POINT LORENTZ TIME DILATION, LENGTH CONTRACTION, AND PROPER TIME

Two well-known consequences of traditional Relativity theory are the lengthening of time intervals (time dilation) and shorting of distances (distance contraction) predicted by Lorentz analytics [2 Chpt. 12, 7 pp. 517, 8 pp. 248 - 250, 9 Sects. 14.0 & 15.0]. In traditional Relativity, Lorentz analytics also defines a combined distance/time "proper time" parameter that has the same value when evaluated in coordinate frames translating relative to one-another [7 pp. 519, 9 Sect. 12]. These effects also arise with Point-to-Point Relativity.

5.1 Point-to-Point Lorentz Time Dilation

Point-to-Point Lorentz time dilation has already been demonstrated in the Section 4.2 two-point solutions where events occurring at one observer were seen by the other observer. When events occurred at point *b*, results in (49) showed that $\Delta t_{a,b1\to b2} = \Delta t_{b,b1\to b2} / \sqrt{1 - v_{ab}^2 / c^2}$, i.e., the time interval measured at point *a* during the spatial movement of point *b* over time instants 1 and 2, was longer than the same time interval measured at point *b*. Similarly, when events occurred at point *a*, results in (53) showed that $\Delta t_{b,a_{1'}\to a_{2'}} = \Delta t_{a,a_{1'}\to a_{2'}} / \sqrt{1 - v_{ab}^2 / c^2}$, i.e., the time interval measured at point *b* during the spatial movement of point *a* over time instants 1' and 2', was longer than the same time interval measured at point *a*. The effect is known as Lorentz "time dilation". The same effect would be obtained for the four and three point solutions when one observer sees distant events at time instant 1 and 2 occurring at the same spatial location and the other sees the events occurring at different times, i.e., when $\underline{x}_{p_1 q_2/b} = 0$ in four-point solution (40), or when $\Delta \underline{x}_{p,1\to 2/b} = 0$ in three-point solution (45). These results are equivalent to what has been obtained with traditional Relativity theory [2 Chpt. 12, 7 pp. 517, 8 pp. 248 - 250, 9 Sect. 14.0].

5.2 <u>Point-to-Point Lorentz Distance Contraction</u>

Point-to-Point distance contraction has been graphically demonstrated in Section 3.4.1 for the particular case when $\sqrt{1 - v_{ab}^2 / c^2} = 0.5$, where distant events *p* and *q* occurred simultaneously relative to observer *b* (but separated in time relative to observer *a*). The Fig. 5a construction then showed that the distance between points *p* and *q* was shorter ("contracted") when measured by observer *b* compared to when measured by observer *a*. The effect can be analytically demonstrated in general from (41) for simultaneous observations of *p* and *q* at *b* (i.e., $\Delta t_{pq/b} = 0$) which finds

$$\Delta t_{pq/a} = \underline{x}_{pq/a} \cdot \underline{v}_{ab/a} / c^2 \tag{55}$$

Substituting (55) in the (41) $\underline{x}_{pq/b}$ equation shows what observer *b* will then find for the distance between events *p* and *q* in terms of observer *a* findings. For the development, it is first expeditious to rearrange the (41) $\underline{x}_{pq/b}$ equation:

$$\begin{aligned}
\underline{x}_{pq/b} &= \underline{x}_{pq/a} - \underline{v}_{ab/a}\,\Delta t_{pq/a} + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(\underline{x}_{pq/a} - \underline{v}_{ab/a}\,\Delta t_{pq/a} \right) \cdot \underline{v}_{ab/a}\,\underline{v}_{ab/a} / v_{ab}^2 \\
&= \underline{x}_{pq/a} - \underline{v}_{ab/a}\,\Delta t_{pq/a} - \left(\underline{x}_{pq/a} - \underline{v}_{ab/a}\,\Delta t_{pq/a} \right) \cdot \underline{v}_{ab/a}\,\underline{v}_{ab/a} / v_{ab}^2 \\
&\quad + \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(\underline{x}_{pq/a} - \underline{v}_{ab/a}\,\Delta t_{pq/a} \right) \cdot \underline{v}_{ab/a}\,\underline{v}_{ab/a} / v_{ab}^2
\end{aligned} \tag{56}$$

Substituting (55) in (56) then obtains

$$
\begin{aligned}
\underline{x}_{pq/b} &= \underline{x}_{pq/a} - \underline{v}_{ab/a}\,\Delta t_{pq/a} - \left(\underline{x}_{pq/a} - \underline{v}_{ab/a}\,\Delta t_{pq/a}\right)\cdot \underline{v}_{ab/a}\,\underline{v}_{ab/a} / v_{ab}^2 \\
&\quad + \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}}\left(\underline{x}_{pq/a} - \underline{v}_{ab/a}\,\Delta t_{pq/a}\right)\cdot \underline{v}_{ab/a}\,\underline{v}_{ab/a} / v_{ab}^2 \\
&= \underline{x}_{pq/a} - \underline{v}_{ab/a}\,\underline{x}_{pq/a}\cdot\underline{v}_{ab/a} / c^2 - \left(\underline{x}_{pq/a} - \underline{v}_{ab/a}\,\underline{x}_{pq/a}\cdot\underline{v}_{ab/a} / c^2\right)\cdot \underline{v}_{ab/a}\,\underline{v}_{ab/a} / v_{ab}^2 \\
&\quad + \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}}\left(\underline{x}_{pq/a}\cdot\underline{v}_{ab/a}\,\underline{v}_{ab/a} / v_{ab}^2 - \underline{v}_{ab/a}\,\underline{x}_{pq/a}\cdot\underline{v}_{ab/a} / c^2\right) \qquad (57) \\
&= \underline{x}_{pq/a} - \underline{x}_{pq/a}\cdot\underline{v}_{ab/a}\,\underline{v}_{ab/a} / v_{ab}^2 + \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}}\left(\frac{1}{v_{ab}^2} - \frac{1}{c^2}\right)\underline{x}_{pq/a}\cdot\underline{v}_{ab/a}\,\underline{v}_{ab/a} \\
&= \underline{x}_{pq/a} - \underline{x}_{pq/a}\cdot\underline{v}_{ab/a}\,\underline{v}_{ab/a} / v_{ab}^2 + \sqrt{1 - v_{ab}^2 / c^2}\;\underline{x}_{pq/a}\cdot\underline{v}_{ab/a}\,\underline{v}_{ab/a} / v_{ab}^2 \\
&= \underline{x}_{pq/a} - \left(1 - \sqrt{1 - v_{ab}^2 / c^2}\right)\underline{x}_{pq/a}\cdot\underline{v}_{ab/a}\,\underline{v}_{ab/a} / v_{ab}^2
\end{aligned}
$$

Equation (57) shows that $\underline{x}_{pq/b}$, the *p* to *q* distance seen by observer *b*, will equal the $\underline{x}_{pq/a}$ observer *a* determined *p* to *q* distance, but with the component parallel to $\underline{v}_{ab/a}$ shortened by the factor $1 - \sqrt{1 - v_{ab}^2 / c^2} \approx \frac{1}{2} v_{ab}^2 / c^2$. The same effect would be seen by observer *a* when $\Delta t_{pq/a} = 0$. Equating the (41) $\Delta t_{pq/a}$ equation to zero finds $\Delta t_{pq/b} = -\underline{x}_{pq/b}\cdot\underline{v}_{ab/b} / c^2$ which, when applied in the (41) $\underline{x}_{pq/a}$ equation, would show that for observer *a*: $\underline{x}_{pq/a} = \underline{x}_{pq/b} - \left(1 - \sqrt{1 - v_{ab}^2 / c^2}\right)\underline{x}_{pq/b}\cdot\underline{v}_{ab/b}\,\underline{v}_{ab/b} / v_{ab}^2$.

The same effect would be obtained for observer *b* in a Section 4.1 type three-point solution when events relative to observer *b* occur simultaneously, i.e., $\Delta t_{b,1\to 2} = 0$ in (45), or when events relative to observer *a* occur simultaneously, i.e., $\Delta t_{a,1\to 2} = 0$ in (45).

The results in this section are equivalent to what has been obtained from traditional Relativity theory [2 Chpt. 12, 7 pp. 517, 8 pp. 248 - 250, 9 Sect. 15.0].

5.3 Point-to-Point Proper Time

In traditional Relativity theory, Lorentz "proper time" is a "time-like" parameter that is invariant in coordinate frames translating relative to one-another [7 pp. 519, 9 pp. Sect. 12.0]. The equivalent for Point-to-Point Relativity derives directly from (46). To expedite the derivation process, it is convenient to reintroduce the $\underline{u}_{\upsilon}$ terminology in (10) for the velocity vector $\underline{v}_{ab/a} = v_{ab}\,\underline{u}_{\upsilon}$. Then with $\underline{v}_{ab/b} = \underline{v}_{ab/a}$ from (12), the first two of (46) become

$$\Delta \underline{x}_{p/a} = \Delta \underline{x}_{p/b} + v_{ab}\, \Delta t_b\, \underline{u}_\upsilon + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(\Delta \underline{x}_{p/b} \cdot \underline{u}_\upsilon + v_{ab}\, \Delta t_b \right) \underline{u}_\upsilon$$
$$\Delta t_a = \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(\Delta t_b + \Delta \underline{x}_{p/b} \cdot \underline{u}_\upsilon\, v_{ab} / c^2 \right) \qquad (58)$$

In the limit as the Δ terms become infinitesimally small, (58) goes to

$$d\underline{x}_{p/a} = d\underline{x}_{p/b} + v_{ab}\, dt_b\, \underline{u}_\upsilon + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(d\underline{x}_{p/b} \cdot \underline{u}_\upsilon + v_{ab}\, dt_b \right) \underline{u}_\upsilon$$
$$dt_a = \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(dt_b + d\underline{x}_{p/b} \cdot \underline{u}_\upsilon\, v_{ab} / c^2 \right) \qquad (59)$$

where

dt_a , dt_b = Differential time intervals measured on the point *a* and *b* clocks.

$d\underline{x}_{p/a}$, $d\underline{x}_{p/b}$ = Differential changes in point *p* position vector over the dt_a , dt_b differential time intervals.

As with traditional Relativity [7 pp. 519, 9 pp. Sect. 12.0], Point-to-Point Relativity proper time is based on its squared value:

$$d\tau^2 \equiv dt^2 - d\underline{x}_p \cdot d\underline{x}_p / c^2 \qquad (60)$$

where

$d\tau$ = Point-to-Point differential proper time interval.

dt = Differential time interval measured on a traditional local clock without particular observer specification (dt_a or dt_b).

$d\underline{x}_p$ = Differential changes in point *p* position vector over the dt differential time interval without particular observer specification ($d\underline{x}_{p/a}$ or $d\underline{x}_{p/b}$).

Note that (60) is similar to the equivalent for traditional Relativity in which proper time is defined as a differential time change function of differential changes in measured distance and time. Similar to traditional Relativity, it will now be shown that Point-to-Point proper time as defined in (60) is the same (i.e., invariant) between observers *a* and *b* translating relative to one another.

For observer a, proper time $d\tau$ calculates from (60) as

$$d\tau^2 = dt_a^2 - d\underline{x}_{p/a} \cdot d\underline{x}_{p/a} / c^2 \tag{61}$$

The differential terms in (61) derive from (59). For the derivation, it is first useful to expand $d\underline{x}_{p/a}$ and $d\underline{x}_{p/b}$ into components parallel and perpendicular to $\underline{u}_\upsilon$:

$$d\underline{x}_{p/a} = d\underline{x}_{p/a_\upsilon} + d\underline{x}_{p/a_\perp} \qquad d\underline{x}_{p/b} = d\underline{x}_{p/b_\upsilon} + d\underline{x}_{p/b_\perp} \tag{62}$$

where

$d\underline{x}_{p/a_\upsilon}$, $d\underline{x}_{p/b_\upsilon}$ = Components of $d\underline{x}_{p/a}$, $d\underline{x}_{p/b}$ parallel to $\underline{u}_\upsilon$.

$d\underline{x}_{pq/a_\perp}$, $d\underline{x}_{p/b_\perp}$ = Components of $d\underline{x}_{p/a}$, $d\underline{x}_{p/b}$ perpendicular to $\underline{u}_\upsilon$.

With (62), (59) becomes

$$\begin{aligned} d\underline{x}_{p/a} &= d\underline{x}_{p/b_\perp} + \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(d\underline{x}_{p/b_\upsilon} + v_{ab}\, dt_b\, \underline{u}_\upsilon \right) \\ dt_a &= \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(dt_b + d\underline{x}_{p/b_\upsilon} \cdot \underline{u}_\upsilon\, v_{ab} / c^2 \right) \end{aligned} \tag{63}$$

Note also that from the definition of the (62) components:

$$\begin{aligned} d\underline{x}_{p/a_\upsilon} &= d\underline{x}_{p/a} \cdot \underline{u}_\upsilon \underline{u}_\upsilon \qquad & d\underline{x}_{p/b_\upsilon} &= d\underline{x}_{p/b} \cdot \underline{u}_\upsilon \underline{u}_\upsilon \\ d\underline{x}_{p/a_\upsilon} \cdot d\underline{x}_{p/a_\upsilon} &= \left(d\underline{x}_{p/a} \cdot \underline{u}_\upsilon \right)^2 \qquad & d\underline{x}_{p/b_\upsilon} \cdot d\underline{x}_{p/b_\upsilon} &= \left(d\underline{x}_{p/b} \cdot \underline{u}_\upsilon \right)^2 \end{aligned} \tag{64}$$

$$\begin{aligned} d\underline{x}_{p/a} \cdot d\underline{x}_{p/a} &= d\underline{x}_{p/a_\upsilon} \cdot d\underline{x}_{p/a_\upsilon} + d\underline{x}_{p/a_\perp} \cdot d\underline{x}_{p/a_\perp} \\ d\underline{x}_{p/b} \cdot d\underline{x}_{p/b} &= d\underline{x}_{p/b_\upsilon} \cdot d\underline{x}_{p/b_\upsilon} + d\underline{x}_{p/b_\perp} \cdot d\underline{x}_{p/b_\perp} \end{aligned} \tag{65}$$

The dt_a^2 and $d\underline{x}_{p/a} \cdot d\underline{x}_{p/a}$ terms in (61) are from (63) with (64) for $\left(d\underline{x}_{p/b_\upsilon} \cdot \underline{u}_\upsilon \right)^2$:

$$d\underline{x}_{p/a} \cdot d\underline{x}_{p/a} =$$

$$\begin{bmatrix} d\underline{x}_{p/b_\perp} \\ +\dfrac{1}{\sqrt{1-v_{ab}^2/c^2}}\begin{pmatrix} d\underline{x}_{p/b_\upsilon} \\ +v_{ab}\,dt_b\,\underline{u}_\upsilon \end{pmatrix} \end{bmatrix} \cdot \begin{bmatrix} d\underline{x}_{p/b_\perp} \\ +\dfrac{1}{\sqrt{1-v_{ab}^2/c^2}}\begin{pmatrix} d\underline{x}_{pq/b_\upsilon} \\ +v_{ab}\,dt_b\,\underline{u}_\upsilon \end{pmatrix} \end{bmatrix}$$

$$= d\underline{x}_{p/b_\perp} \cdot d\underline{x}_{p/b_\perp} + \frac{1}{\left(1-v_{ab}^2/c^2\right)}\begin{bmatrix} d\underline{x}_{p/b_\upsilon} \cdot d\underline{x}_{p/b_\upsilon} \\ +2\,d\underline{x}_{p/b_\upsilon} \cdot \underline{u}_\upsilon\,v_{ab}\,dt_b + v_{ab}^2\,dt_b^2 \end{bmatrix}$$

$$dt_a^2 = \frac{1}{\left(1-v_{ab}^2/c^2\right)}\left(dt_b + d\underline{x}_{p/b_\upsilon} \cdot \underline{u}_\upsilon\,v_{ab}/c^2\right)^2 \tag{66}$$

$$= \frac{1}{\left(1-v_{ab}^2/c^2\right)}\left[dt_b^2 + 2\,d\underline{x}_{p/b_\upsilon} \cdot \underline{u}_\upsilon\,v_{ab}\,dt_b/c^2 + \left(d\underline{x}_{p/b_\upsilon} \cdot \underline{u}_\upsilon\right)^2 v_{ab}^2/c^4\right]$$

Substituting (66) in (61) then finds for Point-to-Point Relativity proper time:

$$d\tau^2 = \frac{1}{\left(1-v_{ab}^2/c^2\right)}\left[dt_b^2 + 2\,d\underline{x}_{p/b_\upsilon} \cdot \underline{u}_\upsilon\,v_{ab}\,dt_{pq/b}/c^2 + d\underline{x}_{p/b_\upsilon} \cdot d\underline{x}_{p/b_\upsilon}\,v_{ab}^2/c^4\right]$$

$$- d\underline{x}_{p/b_\perp} \cdot d\underline{x}_{p/b_\perp}/c^2$$

$$- \frac{1}{\left(1-v_{ab}^2/c^2\right)}\left[d\underline{x}_{p/b_\upsilon} \cdot d\underline{x}_{p/b_\upsilon} + 2\,d\underline{x}_{p/b_\upsilon} \cdot \underline{u}_\upsilon\,v_{ab}\,dt_b + v_{ab}^2\,dt_b^2\right]/c^2 \tag{67}$$

$$= \frac{1}{\left(1-v_{ab}^2/c^2\right)}\left[\left(1-v_{ab}^2/c^2\right)dt_b^2 + \left(v_{ab}^2/c^4 - 1/c^2\right)d\underline{x}_{p/b_\upsilon} \cdot d\underline{x}_{p/b_\upsilon}\right]$$

$$- d\underline{x}_{p/b_\perp} \cdot d\underline{x}_{p/b_\perp}/c^2$$

$$= \Delta t_b^2 - \left(d\underline{x}_{p/b_\upsilon} \cdot d\underline{x}_{p/b_\upsilon} - d\underline{x}_{p/b_\perp} \cdot d\underline{x}_{p/b_\perp}\right)/c^2$$

or with (65) and (61):

$$d\tau^2 = dt_a^2 - d\underline{x}_{p/a} \cdot d\underline{x}_{p/a}/c^2 = dt_b^2 - d\underline{x}_{p/b} \cdot d\underline{x}_{p/b}/c^2 \tag{68}$$

Equation (68) demonstrates the invariance of Point-to-Point proper time formula (60) as determined by observer *a* or by observer *b*. The (68) results are equivalent to what has been obtained with traditional Relativity theory [7 pp. 519, 9 Sect. 12.0].

Equation (68) can also be used to show the relationship between proper time and the time differential measured on the point *a* and *b* clocks. From (68),

$$d\tau^2 = dt_a^2 - d\underline{x}_{p/a} \cdot d\underline{x}_{p/a} / c^2 = \left(1 - \frac{d\underline{x}_{p/a}}{dt_a} \cdot \frac{d\underline{x}_{p/a}}{dt_a} / c^2\right) dt_a^2 \qquad (69)$$

Then, from (69) and the equivalent for the observer b time differential:

$$\begin{aligned} dt_a = d\tau / \sqrt{1 - \underline{v}_{p/a} \cdot \underline{v}_{p/a} / c^2} \qquad & dt_b = d\tau / \sqrt{1 - \underline{v}_{p/b} \cdot \underline{v}_{p/b} / c^2} \\ \underline{v}_{p/a} \equiv \frac{d\underline{x}_{p/a}}{dt_a} \qquad & \underline{v}_{p/b} \equiv \frac{d\underline{x}_{p/b}}{dt_b} \end{aligned} \qquad (70)$$

where

$\underline{v}_{p/a}$, $\underline{v}_{p/b}$ = Velocity vector of point p as determined by observers a and b.

Equations (70) also show that:

$$\frac{dt_a}{dt_b} = \sqrt{\frac{\left(1 - \underline{v}_{p/b} \cdot \underline{v}_{p/b} / c^2\right)}{\left(1 - \underline{v}_{p/a} \cdot \underline{v}_{p/a} / c^2\right)}} \qquad (71)$$

6.0 POINT-TO-POINT DERIVATIVE FORMS

The analysis thus far has concentrated on formulations of relative distance measurements from the viewpoint of two observers traveling relative to one-another. In this section, the moving point p approach in Section 4.1 is extended to encompass a and b point observations of point p velocity and acceleration:

$$\underline{v}_{p/a} \equiv \frac{d\underline{x}_{p/a}}{dt_a} \qquad \underline{a}_{p/a} \equiv \frac{d\underline{v}_{p/a}}{dt_a} \qquad \underline{v}_{p/b} \equiv \frac{d\underline{x}_{p/b}}{dt_b} \qquad \underline{a}_{pq/b} \equiv \frac{d\underline{v}_{p/b}}{dt_b} \qquad (72)$$

where

$\underline{v}_{p/a}$, $\underline{v}_{p/b}$ = Velocity vector of point p as determined by observers a and b.

$\underline{a}_{p/a}$, $\underline{a}_{p/b}$ = Change in the point p velocity vector per differential time interval (i.e., the acceleration of point p), as determined by observers a and b.

6.1 Velocity

The derivation of point p velocity relative to point a begins with a restatement of (63):

$$
\begin{aligned}
d\underline{x}_{p/a} &= d\underline{x}_{p/b_\perp} + \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(d\underline{x}_{p/b_\upsilon} + v_{ab}\,dt_b\,\underline{u}_\upsilon\right) \\
dt_a &= \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(dt_b + d\underline{x}_{p/b_\upsilon}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)
\end{aligned}
\tag{73}
$$

Dividing (73) by dt_b finds

$$
\begin{aligned}
\frac{d\underline{x}_{p/a}}{dt_b} &= \frac{d\underline{x}_{p/b_\perp}}{dt_b} + \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(\frac{d\underline{x}_{p/b_\upsilon}}{dt_b} + v_{ab}\,\underline{u}_\upsilon\right) \\
\frac{dt_a}{dt_b} &= \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(1 + \frac{d\underline{x}_{p/b_\upsilon}}{dt_b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)
\end{aligned}
\tag{74}
$$

From (62) and (64):

$$
d\underline{x}_{p/b_\upsilon} \equiv d\underline{x}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon \qquad d\underline{x}_{p/b_\perp} = d\underline{x}_{p/b} - d\underline{x}_{p/b_\upsilon} = d\underline{x}_{p/b} - d\underline{x}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon \tag{75}
$$

so that with (72):

$$
\begin{aligned}
\frac{d\underline{x}_{p/b_\upsilon}}{dt_b} &= \frac{d\underline{x}_{p/b}}{dt_b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon = \underline{v}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon \\
\frac{d\underline{x}_{p/b_\perp}}{dt_b} &= \frac{d\underline{x}_{p/b}}{dt_b} - \frac{d\underline{x}_{p/b}}{dt_b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon = \underline{v}_{p/b} - \underline{v}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon
\end{aligned}
\tag{76}
$$

Substituting (76) in (74) then obtains

$$
\begin{aligned}
\frac{d\underline{x}_{p/a}}{dt_b} &= \underline{v}_{p/b} - \underline{v}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon + \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(\underline{v}_{p/b}\cdot\underline{u}_\upsilon + v_{ab}\right)\underline{u}_\upsilon \\
\frac{dt_a}{dt_b} &= \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(1 + \underline{v}_{p/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)
\end{aligned}
\tag{77}
$$

Trivially, from (72):

$$
\underline{v}_{p/a} = \frac{d\underline{x}_{p/a}}{dt_a} = \frac{d\underline{x}_{p/a}}{dt_b}\,\frac{dt_b}{dt_a} = \left(\frac{dt_a}{dt_b}\right)^{-1}\frac{d\underline{x}_{p/a}}{dt_b} \tag{78}
$$

Substituting (77) into (78) yields

$$\underline{v}_{p/a} = \frac{\sqrt{1 - v_{ab}^2 / c^2}\left(\underline{v}_{p/b} - \underline{v}_{p/b} \cdot \underline{u}_\upsilon \, \underline{u}_\upsilon\right)}{\left(1 + \underline{v}_{p/b} \cdot \underline{u}_\upsilon \, v_{ab} / c^2\right)} + \frac{\left(\underline{v}_{p/b} \cdot \underline{u}_\upsilon + v_{ab}\right)\underline{u}_\upsilon}{\left(1 + \underline{v}_{p/b} \cdot \underline{u}_\upsilon \, v_{ab} / c^2\right)} \tag{79}$$

or since $\underline{u}_\upsilon = \underline{v}_{ab/b} / v_{ab}$ from (9) and (12):

$$\underline{v}_{p/a} = \frac{\sqrt{1 - v_{ab}^2 / c^2}\left(\underline{v}_{p/b} - \underline{v}_{p/b} \cdot \underline{v}_{ab/b} \, \underline{v}_{ab/b} / v_{ab}^2\right)}{\left(1 + \underline{v}_{p/b} \cdot \underline{v}_{ab/b} / c^2\right)} + \frac{\left(1 + \underline{v}_{p/b} \cdot \underline{v}_{ab/b} / v_{ab}^2\right)\underline{v}_{ab/b}}{\left(1 + \underline{v}_{p/b} \cdot \underline{v}_{ab/b} / c^2\right)} \tag{80}$$

From the equivalent (79) form of (80) it should be apparent that the second term in (80) is parallel to $\underline{u}_\upsilon$ (i.e., parallel to $\underline{v}_{ab/b}$), and the first term is perpendicular to $\underline{v}_{ab/b}$ (i.e., $\underline{v}_{p/b}$ minus the $\underline{v}_{p/b} \cdot \underline{u}_\upsilon \, \underline{u}_\upsilon = \underline{v}_{p/b} \cdot \underline{v}_{ab/b} \, \underline{v}_{ab/b} / v_{ab}^2$ parallel component).

The (80) result is equivalent to what has previously been obtained using traditional Relativity theory [7 pp. Eq. (12-12), 9 pp. Eq. (16.07)]. Similar results can be formulated for $\underline{v}_{p/b}$ components in terms of $\underline{v}_{p/a}$ and $\underline{v}_{ab/a}$.

6.2 Acceleration

The derivation of the point *p* acceleration formula begins with (79), the $\frac{dt_a}{dt_b}$ equation in (77), and the acceleration definitions in (72). Taking the derivative of (79) (with $\underline{v}_{ab}$ constant for this article), and applying definitions (72) yields

$$\begin{aligned}\frac{d\underline{v}_{p/a}}{dt_b} &= \frac{\sqrt{1 - v_{ab}^2 / c^2}\left(\underline{a}_{p/b} - \underline{a}_{p/b} \cdot \underline{u}_\upsilon \, \underline{u}_\upsilon\right)}{\left(1 + \underline{v}_{p/b} \cdot \underline{u}_\upsilon \, \underline{v}_{ab} / c^2\right)} \\ &- \frac{\sqrt{1 - v_{ab}^2 / c^2}\left(\underline{v}_{p/b} - \underline{v}_{p/b} \cdot \underline{u}_\upsilon \, \underline{u}_\upsilon\right)\underline{a}_{p/b} \cdot \underline{u}_\upsilon \, v_{ab} / c^2}{\left(1 + \underline{v}_{p/b} \cdot \underline{u}_\upsilon \, v_{ab} / c^2\right)^2} + \frac{\underline{a}_{p/b} \cdot \underline{u}_\upsilon \, \underline{u}_\upsilon}{\left(1 + \underline{v}_{p/b} \cdot \underline{u}_\upsilon \, v_{ab} / c^2\right)} \\ &- \frac{\left(\underline{v}_{p/b} \cdot \underline{u}_\upsilon \, \underline{u}_\upsilon + v_{ab} \, \underline{u}_\upsilon\right)\underline{a}_{p/b} \cdot \underline{u}_\upsilon \, v_{ab} / c^2}{\left(1 + \underline{v}_{p/b} \cdot \underline{u}_\upsilon \, v_{ab} / c^2\right)^2}\end{aligned} \tag{81}$$

Trivially, with (72):

$$\underline{a}_{p/a} = \frac{d\underline{v}_{p/a}}{dt_a} = \frac{d\underline{v}_{p/a}}{dt_b}\,\frac{dt_b}{dt_a} = \left(\frac{dt_a}{dt_b}\right)^{-1}\frac{d\underline{v}_{p/a}}{dt_b} \tag{82}$$

Substituting (81) and the (77) $\frac{dt_a}{dt_b}$ expression into (82) then yields

$$\begin{aligned}\underline{a}_{p/a} &= \frac{\left(1-v_{ab}^2/c^2\right)\left(\underline{a}_{p/b}-\underline{a}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon\right)}{\left(1+\underline{v}_{p/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)^2} \\ &-\frac{\left(1-v_{ab}^2/c^2\right)\left(\underline{v}_{p/b}-\underline{v}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon\right)\underline{a}_{p/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2}{\left(1+\underline{v}_{p/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)^3}+\frac{\sqrt{1-v_{ab}^2/c^2}\;\underline{a}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon}{\left(1+\underline{v}_{p/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)^2} \\ &-\frac{\sqrt{1-v_{ab}^2/c^2}\left(\underline{v}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon+v_{ab}\,\underline{u}_\upsilon\right)\underline{a}_{p/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2}{\left(1+\underline{v}_{p/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)^3}\end{aligned} \tag{83}$$

The last two terms in (83) combine as follows:

$$\begin{aligned}&\frac{\sqrt{1-v_{ab}^2/c^2}\;\underline{a}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon}{\left(1+\underline{v}_{p/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)^2}-\frac{\sqrt{1-v_{ab}^2/c^2}\left(\underline{v}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon+v_{ab}\,\underline{u}_\upsilon\right)\underline{a}_{p/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2}{\left(1+\underline{v}_{p/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)^3} \\ &= \frac{\sqrt{1-v_{ab}^2/c^2}}{\left(1+\underline{v}_{pq/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)^3}\left[\begin{array}{c}\left(1+\underline{v}_{p/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)\underline{a}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon \\ -\left(\underline{v}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon+v_{ab}\,\underline{u}_\upsilon\right)\underline{a}_{p/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\end{array}\right] \\ &= \frac{\sqrt{1-v_{ab}^2/c^2}}{\left(1+\underline{v}_{p/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)^3}\left[\begin{array}{c}\left(1-v_{ab}^2/c^2\right)\underline{a}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon \\ +\underline{v}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon\left(\underline{a}_{p/b}\cdot\underline{u}_\upsilon v_{ab}/c^2-\underline{a}_{p/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)\end{array}\right] \\ &= \frac{\left(1-v_{ab}^2/c^2\right)^{3/2}\underline{a}_{p/b}\cdot\underline{u}_\upsilon\,\underline{u}_\upsilon}{\left(1+\underline{v}_{pq/b}\cdot\underline{u}_\upsilon\, v_{ab}/c^2\right)^3}\end{aligned} \tag{84}$$

with which (83) then becomes

$$\underline{a}_{p/a} = \frac{\left(1 - v_{ab}^2/c^2\right)^{3/2} \underline{a}_{p/b} \cdot \underline{u}_\upsilon \underline{u}_\upsilon}{\left(1 + \underline{v}_{p/b} \cdot \underline{v}_{ab}/c^2\right)^3} + \frac{\left(1 - v_{ab}^2 / c^2\right)\left(\underline{a}_{p/b} - \underline{a}_{p/b} \cdot \underline{u}_\upsilon \underline{u}_\upsilon\right)}{\left(1 + \underline{v}_{p/b} \cdot \underline{v}_{ab}/c^2\right)^2}$$
$$- \frac{\left(1 - v_{ab}^2 / c^2\right)\left(\underline{v}_{p/b} - \underline{v}_{p/b} \cdot \underline{u}_\upsilon \underline{u}_\upsilon\right) \underline{a}_{p/b} \cdot \underline{u}_\upsilon v_{ab} / c^2}{\left(1 + \underline{v}_{p/b} \cdot \underline{v}_{ab} / c^2\right)^3} \qquad (85)$$

or since $\underline{u}_\upsilon = \underline{v}_{ab/b} / v_{ab}$ from (9) and (12):

$$\underline{a}_{p/a} = \frac{\left(1 - v_{ab}^2/c^2\right)^{3/2} \underline{a}_{p/b} \cdot \underline{v}_{ab/b} \underline{v}_{ab/b} / v_{ab}^2}{\left(1 + \underline{v}_{p/b} \cdot \underline{v}_{ab/b}/c^2\right)^3} + \frac{\left(1 - v_{ab}^2 / c^2\right)\left(\underline{a}_{p/b} - \underline{a}_{p/b} \cdot \underline{v}_{ab/b} \underline{v}_{ab/b} / v_{ab}^2\right)}{\left(1 + \underline{v}_{p/b} \cdot \underline{v}_{ab/b}/c^2\right)^2}$$
$$- \frac{\left(1 - v_{ab}^2 / c^2\right)\left(\underline{v}_{p/b} - \underline{v}_{p/b} \cdot \underline{v}_{ab/b} \underline{v}_{ab/b} / v_{ab}^2\right) \underline{a}_{p/b} \cdot \underline{v}_{ab/b} / c^2}{\left(1 + \underline{v}_{p/b} \cdot \underline{v}_{ab/b} / c^2\right)^3} \qquad (86)$$

From the equivalent (85) form of (86) it should be apparent that the first term in (86) is parallel to $\underline{u}_\upsilon$ (i.e., parallel to $\underline{v}_{ab/b}$), the second term is perpendicular to $\underline{v}_{ab/b}$ (i.e., $\underline{a}_{p/b}$ minus the $\underline{a}_{p/b} \cdot \underline{u}_\upsilon \underline{u}_\upsilon = \underline{a}_{p/b} \cdot \underline{v}_{ab/b} \underline{v}_{ab/b} / v_{ab}^2$ parallel component), and as in the previous section, the third term is also perpendicular to $\underline{v}_{ab/b}$ (i.e., $\underline{v}_{p/b}$ minus the component parallel to $\underline{v}_{ab/b}$: $\underline{v}_{p/b} \cdot \underline{u}_\upsilon \underline{u}_\upsilon = \underline{v}_{p/b} \cdot \underline{v}_{ab/b} \underline{v}_{ab/b} / v_{ab}^2$).

The (86) result is equivalent to what has been obtained previously based on traditional Relativity theory [7 Eq. (12-13)]. Similar results can be formulated for $\underline{a}_{p/b}$ components in terms of $\underline{a}_{p/a}$ and $\underline{v}_{ab/a}$.

7.0 CONCLUSIONS

Point-to-Point Relativity kinematics provide an analytical equivalent to Lorentz transformation operations in modern Relativity Theory. The analytical notation developed for the Point-to-Point approach enables straight-forward development of relativistic formulas, with clear definitions for time and distance parameters, and without the use of space-time diagrams, world lines intersecting space-time events, and the concept of space-time simultaneity prevalent in traditional Relativity formulations.

Traditional Relativity theory is based on the comparative observation of remote events in coordinate frames that are translating relative to one another, and in which position locations are defined relative to the origin of the translating frames. In the Point-to-Point formulation, remote phenomena are observed at two spatial points that translate relative to one another, and in which position locations are defined relative to the observer location points. Coordinate frames are only used as a means for determining the components of vector parameters as projected on specified coordinate frame axes.

The basic Point-to-Point Lorentz conversion formulas define how the relative distance vector between space-time points measured by one observer are related to the same distance vector as measured by a second observer in motion relative to the first. The relative distance formulas can be used to demonstrate relativistic time dilation and distance contraction of observed remote events, define a "proper time" interval that is invariant to any observer, and how relative velocity and acceleration vectors between space-time points measured by one observer are related to the same vectors measured by a second observer. Overall results match their equivalents based on traditional Relativity theory.

All Point-to-Point Relativity kinematic results presented in this article derive from basic Eqs. (3), (8), (13), (18), (19), and (37).

APPENDIX A - NON-ROTATING INERTIAL COORDINATES

Point-to-Point kinematics is presented in this article in vector format for vector component projection on so-called "non-rotating inertial coordinate frame" axes, i.e., as traditionally defined: "coordinate frames in which Newtonian and Relativity theory is valid". A measurable definition of non-rotating inertial coordinates is presented in [6] based on a redefined version of Newton's dynamic law of motion:

$$\frac{d^2 \underline{x}_{i\,j}}{dt^2} = \underline{a}_{F/j} - \underline{a}_{F/i} + \Delta \underline{g}_{i\,j} \tag{A-1}$$

where $\underline{x}_{i\,j}$ (as in this article) is the distance vector from mass point i to mass point j, $\frac{d^2 \underline{x}_{i\,j}}{dt^2}$ is the vector acceleration of mass point j relative to mass point i, $\underline{a}_{F/j}$ and $\underline{a}_{F/i}$ are local force generated vector accelerations of mass points i and j (sometimes denoted as "specific force") measurable by accelerometers located at points i and j, and $\Delta \underline{g}_{i\,j}$ is the difference in gravity at point j relative to point i (created by the distributed mass of the universe).

An accelerometer is a device that directly measures force-generated acceleration using a proof mass located in a body whose force acceleration is to be measured. The proof-mass position location is controlled by forces generated within the accelerometer to maintain a fixed location of the proof-mass within the body-mounted accelerometer case. The resulting proof-mass force acceleration is thereby controlled to equal the body's force-generated acceleration. By dividing the measured accelerometer control force by the mass of the proof mass, a direct measurement of body force acceleration is obtained. Most accelerometers are designed to measure force acceleration along a single axis (the accelerometer "input axis"). Three accelerometers are then required to measure the three components of each of the force acceleration vectors, $\underline{a}_{F/j}$ and $\underline{a}_{F/i}$ in (A-1).

Newton's law states that (A-1) is valid in inertial non-rotating frames [1, pp. 416]. Thus, if arbitrary coordinate frame I is inertially non-rotating, the vector/matrix form of (A-1) using Appendix B notation, would be

$$\frac{d^2\underline{x}_{ij}^{\mathrm{I}}}{dt^2} = \underline{a}_{F/j}^{\mathrm{I}} - \underline{a}_{F/i}^{\mathrm{I}} + \Delta\underline{g}_{ij}^{\mathrm{I}} \qquad \text{(A-2)}$$

where

$\underline{x}_{ij}^{\mathrm{I}}$ = Column matrix with elements equal to the projections of $\underline{x}_{ij}$ on I frame axes.

$\frac{d^2\underline{x}_{ij}^{\mathrm{I}}}{dt^2}$ = Column matrix with elements equal to the second derivative of the I frame component projections of $\underline{x}_{ij}$ on coordinate frame I axes.

In contrast, reference [6] shows that for $\underline{x}_{ij}$ projection on the axes of coordinate frame B that is rotating relative to frame I, (A-2) expands to:

$$\frac{d^2\underline{x}_{ij}^{\mathrm{B}}}{dt^2} = \underline{a}_{F/j}^{\mathrm{B}} - \underline{a}_{F/i}^{\mathrm{B}} + \Delta\underline{g}_{ij}^{\mathrm{B}} - \frac{d\underline{\omega}_{\mathrm{IB}}^{\mathrm{B}}}{dt} \times \underline{x}_{ij}^{\mathrm{B}} - \underline{\omega}_{\mathrm{IB}}^{\mathrm{B}} \times \left(\underline{\omega}_{\mathrm{IB}}^{\mathrm{B}} \times \underline{x}_{ij}^{\mathrm{B}}\right) - 2\,\underline{\omega}_{\mathrm{IB}}^{\mathrm{B}} \times \frac{d\underline{x}_{ij}^{\mathrm{B}}}{dt} \qquad \text{(A-3)}$$

where

$\underline{\omega}_{\mathrm{IB}}^{\mathrm{B}}$ = "Inertial angular rate" of coordinate frame B relative to inertially non-rotating coordinate frame I (IB subscript) as projected on rotating frame B axes (B superscript).

$\frac{d^2\underline{x}_{ij}^{\mathrm{B}}}{dt^2}$ = Column matrix with elements equal to the second derivative of the B frame component projections of $\underline{x}_{ij}$ on coordinate frame B axes.

Comparing (A-3) with (A-2), the appended last three terms in (A-3) have been denoted respectively as tangential acceleration caused by angular acceleration, centripetal acceleration, and Coriolis acceleration. When $\underline{\omega}_{\mathrm{IB}}^{\mathrm{B}} = 0$ over time, coordinate frame B becomes inertially non-rotating for which from (A-3):

$$\frac{d^2 \underline{x}_{ij}^{\mathrm{B}}}{dt^2} = \underline{a}_{F/j}^{\mathrm{B}} - \underline{a}_{F/i}^{\mathrm{B}} + \Delta \underline{g}_{ij}^{\mathrm{B}} \tag{A-4}$$

which is of exactly the same form as (A-2). Thus, to determine whether a particular B frame is inertially non-rotating, measure $\underline{a}_{F/j}^{\mathrm{B}} - \underline{a}_{F/i}^{\mathrm{B}}$ using accelerometers; then add $\Delta \underline{g}_{ij}^{\mathrm{B}}$. If the result equals $\frac{d^2 \underline{x}_{ij}^{\mathrm{B}}}{dt^2}$ measured by non-inertial means (e.g., by e/m transmission/reception as in Appendix C), the B frame is inertially non-rotating. These results can also be used to define a non-rotating inertial frame of reference as being a coordinate frame in which $\frac{d^2 \underline{x}_{ij}^{\mathrm{B}}}{dt^2}$ measured by non-inertial means satisfies (A-4).

This article only deals with Point-to-Point vector kinematic projections on inertially non-rotating coordinate frames. Planned future articles will analyze Point-to-Point kinematics projections on inertially rotating coordinates.

The previous equations are based on redefined Newtonian dynamic theory, hence, do not include the effects of Relativity on measurements by observers at different locations, or the impact of Relativity theory on gravity. This article only deals with the kinematics of Point-to-Point Relativity, not on the dynamics of velocity change produced by applied local forces and gravity as in (A-1) and (A-3). Future articles will expand on these results to account for dynamic changes in the relative motion of mass points created by measurable locally applied forces ($\underline{a}_{F/j}$ and $\underline{a}_{F/i}$), and for $\Delta \underline{g}_{ij}$ gravity differences between mass point locations.

APPENDIX B - VECTORS AND COORDINATE FRAMES

As used in this article, a vector is a three-dimensional parameter that has length and direction. Vectors in this article are classified as “free vectors”, i.e., having no preferred location in coordinate frames in which they are analytically described. A coordinate frame is defined in this article as an analytical abstraction represented by three mutually orthogonal free vectors of unity magnitude. The components of a vector in a particular coordinate frame equal the dot product of the vector with its coordinate frame unit vectors. The physical position location of each coordinate frame’s origin is arbitrary.

Based on the previous definitions, the component projection of an arbitrary vector $\underline{W}$ in an arbitrary coordinate frame A would be

$$
\begin{array}{lll}
W_{XA} = \underline{W} \cdot \underline{u}_{XA} & W_{YA} = \underline{W} \cdot \underline{u}_{YA} & W_{ZA} = \underline{W} \cdot \underline{u}_{ZA} \\
\underline{u}_{XA} \cdot \underline{u}_{XA} = 1 & \underline{u}_{XA} \cdot \underline{u}_{YA} = 0 & \underline{u}_{XA} \cdot \underline{u}_{ZA} = 0 \\
\underline{u}_{YA} \cdot \underline{u}_{XA} = 0 & \underline{u}_{YA} \cdot \underline{u}_{YA} = 1 & \underline{u}_{YA} \cdot \underline{u}_{ZA} = 0 \\
\underline{u}_{ZA} \cdot \underline{u}_{XA} = 0 & \underline{u}_{ZA} \cdot \underline{u}_{YA} = 0 & \underline{u}_{ZA} \cdot \underline{u}_{ZA} = 1
\end{array}
\qquad \text{(B-1)}
$$

where

$\underline{u}_{XA}$, $\underline{u}_{YA}$, $\underline{u}_{ZA}$ = Mutually orthogonal unit vectors associated with coordinate frame A.

W_{XA}, W_{YA}, W_{ZA} = Components of vector $\underline{W}$ "along coordinate frame A axes" or "in A frame coordinates".

Based on (B-1), vector $\underline{W}$ can then expressed as the sum of its A frame component projections:

$$
\underline{W} = W_{XA}\, \underline{u}_{XA} + W_{YA}\, \underline{u}_{YA} + W_{ZA}\, \underline{u}_{ZA} \qquad \text{(B-2)}
$$

The same vector $\underline{W}$ can also be expressed in terms of its projections along the axes of another arbitrary coordinate frame B:

$$
\underline{W} = W_{XB}\, \underline{u}_{XB} + W_{YB}\, \underline{u}_{YB} + W_{ZB}\, \underline{u}_{ZB} \qquad \text{(B-3)}
$$

where

$\underline{u}_{XB}$, $\underline{u}_{YB}$, $\underline{u}_{ZB}$ = Mutually orthogonal unit vectors associated with coordinate frame B.

W_{XB}, W_{YB}, W_{ZB} = Components of vector $\underline{W}$ along coordinate frame B axes.

with as in (B-1):

$$
\begin{array}{lll}
W_{XB} = \underline{W} \cdot \underline{u}_{XB} & W_{YB} = \underline{W} \cdot \underline{u}_{YB} & W_{ZB} = \underline{W} \cdot \underline{u}_{ZB} \\
\underline{u}_{XB} \cdot \underline{u}_{XB} = 1 & \underline{u}_{XB} \cdot \underline{u}_{YB} = 0 & \underline{u}_{XB} \cdot \underline{u}_{ZB} = 0 \\
\underline{u}_{YB} \cdot \underline{u}_{XB} = 0 & \underline{u}_{YB} \cdot \underline{u}_{YB} = 1 & \underline{u}_{YB} \cdot \underline{u}_{ZB} = 0 \\
\underline{u}_{ZB} \cdot \underline{u}_{XB} = 0 & \underline{u}_{ZB} \cdot \underline{u}_{YB} = 0 & \underline{u}_{ZB} \cdot \underline{u}_{ZB} = 1
\end{array}
\qquad \text{(B-4)}
$$

An expression for the A frame component of $\underline{W}$ in terms of the B frame components can be found by taking the dot product of (B-2) with the B frame unit vectors:

$$
\begin{aligned}
W_{XA} &= \underline{u}_{XA} \cdot \underline{u}_{XB}\, W_{XB} + \underline{u}_{XA} \cdot \underline{u}_{YB}\, W_{YB} + \underline{u}_{XA} \cdot \underline{u}_{ZB}\, W_{ZB} \\
W_{YA} &= \underline{u}_{YA} \cdot \underline{u}_{XB}\, W_{XB} + \underline{u}_{YA} \cdot \underline{u}_{YB}\, W_{YB} + \underline{u}_{YA} \cdot \underline{u}_{ZB}\, W_{ZB} \\
W_{ZA} &= \underline{u}_{ZA} \cdot \underline{u}_{XB}\, W_{XB} + \underline{u}_{ZA} \cdot \underline{u}_{YB}\, W_{YB} + \underline{u}_{ZA} \cdot \underline{u}_{ZB}\, W_{ZB}
\end{aligned}
\qquad \text{(B-5)}
$$

The frame B components as a function of the frame A components are obtained similarly using dot products of (B-3) with the A frame unit vectors:

$$
\begin{aligned}
W_{XB} &= \underline{u}_{XB} \cdot \underline{u}_{XA} W_{XA} + \underline{u}_{XB} \cdot \underline{u}_{YA} W_{YA} + \underline{u}_{XB} \cdot \underline{u}_{ZA} W_{ZA} \\
W_{YB} &= \underline{u}_{YB} \cdot \underline{u}_{XA} W_{XA} + \underline{u}_{YB} \cdot \underline{u}_{YA} W_{YA} + \underline{u}_{YB} \cdot \underline{u}_{ZA} W_{ZA} \qquad \text{(B-6)} \\
W_{ZB} &= \underline{u}_{ZB} \cdot \underline{u}_{XA} W_{XA} + \underline{u}_{ZB} \cdot \underline{u}_{YA} W_{YA} + \underline{u}_{ZB} \cdot \underline{u}_{ZA} W_{ZA}
\end{aligned}
$$

Because $\underline{u}_{iB}$ and $\underline{u}_{jA}$ are unit vectors, $\underline{u}_{iB} \cdot \underline{u}_{jA}$ in (B-5) and (B-6) (i and j being X, Y, or Z) equals the cosine of the angle between $\underline{u}_{iB}$ and $\underline{u}_{jA}$. Thus, (B-5) and (B-6) can also be expressed using the matrix notation introduced by Britting in [10]:

$$
\underline{W}^{A} = C_{B}^{A}\, \underline{W}^{B} \qquad \underline{W}^{B} = C_{A}^{B}\, \underline{W}^{A}
$$

$$
\underline{W}^{A} \equiv \begin{bmatrix} W_{XA} \\ W_{YA} \\ W_{ZA} \end{bmatrix} \quad \underline{W}^{B} \equiv \begin{bmatrix} W_{XB} \\ W_{YB} \\ W_{ZB} \end{bmatrix} \quad C_{B}^{A} \equiv \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \quad C_{B}^{A} \equiv \begin{bmatrix} \gamma_{11} & \gamma_{21} & \gamma_{31} \\ \gamma_{12} & \gamma_{22} & \gamma_{32} \\ \gamma_{13} & \gamma_{23} & \gamma_{33} \end{bmatrix} \qquad \text{(B-7)}
$$

where

$\underline{W}^{A}$, $\underline{W}^{B}$ = Column matrices with elements equal to the components of $\underline{W}$ projected on ("along") A frame and B frame axes.

C_{B}^{A}, C_{A}^{B} = Direction cosine matrices that transform vectors from their A frame matrix form to their B frame matrix form, and from their B frame matrix form to their A frame matrix form.

γ_{ij} = Cosine of the angle between A frame unit vector i and B frame unit vector j; i and j from 1 to 3 corresponding to X, Y, and Z in (B-5) and (B-6).

Note from (B-7) that C_{A}^{B} is the matrix transpose of C_{B}^{A}.

APPENDIX C - MEASURING RELATIVE DISTANCE AND VELOCITY BY ELECTRO-MAGNETIC WAVE TRANSMISSION/REFLECTION

Distances between two points can be determined by measuring the time for a wave of electro-magnetic (e/m) radiation to travel from one point to the other, divided by the velocity of e/m wave propagation through space. Past experiments have demonstrated that relative to any observer, all e/m waves, regardless of wavelength and original source velocity, propagate through open space at the same speed (i.e., at the "speed of light"). Using this principle, range-radars, and recently laser radars ("ladars"), have been commonly used to measure the range to distant targets. For a radar or ladar receiving antenna that is stabilized relative to non-rotating inertial space, the angular direction of the received e/m wave can also be determined, hence, the direction vector from the antenna to the target. Combining the antenna-to-target distance and direction vector determines the distance vector from the antenna to the target.

C.1 Target Distance And Time Instant At Target Illumination

For a target at point *p* and a radar transmitter at point *a*, consider that a radar e/m pulse is transmitted at time instant 1, reflected from point *p* at time point 2, and the reflection received at point *a* at time instant 3. (Note: The 1, 2, 3 time instant notation used in this appendix is different from the time instant notation used in the main article). The range from *a* to *p* and time instant 2 at target reflection would then be determined at point *a* from

$$\Delta t_{a,1\to 2} = \left(t_{a,3} - t_{a,1}\right)/2 = \Delta t_{a,1\to 3}/2 \qquad \text{(C-1)}$$

$$x_{ap,2/a} = c\,\Delta t_{a,1\to 2} = c\,\Delta t_{a,1\to 3}/2 \qquad \text{(C-2)}$$

$$\begin{aligned}\Delta t_{a,1\to 2} &= t_{a,1} + \Delta t_{a,1\to 3}/2 \\ &= t_{a,1} + \left(t_{a,3} - t_{a,1}\right)/2 = \left(t_{a,3} + t_{a,1}\right)/2\end{aligned} \qquad \text{(C-3)}$$

where

c = The propagation velocity of electro-magnetic (e/m) waves through open space (the "speed of light").

$t_{a,1}$ = Time instant 1 on a clock at point *a* when the e/m wave pulse was emitted.

$t_{a,3}$ = Time instant 3 when the reflected e/m wave pulse at point *p* is received back at point *a*.

$\Delta t_{a,1\to 3}$ = Elapsed time from $t_{a,1}$ to $t_{a,3}$ on the point *a* clock, as observed at point *a*.

$x_{ap,2/a}$ = Range (distance) from *a* to *p* when the e/m pulse arrived at *p* at time instant $t_{a,2}$. Note: This distance is not measured directly, but is calculated at point *a* based on the $\Delta t_{a,1\to 3}$ measurement.

$t_{a,2}$ = Time instant 2 when the e/m pulse from point *a* reached target *p*. Note: This time instant is not measured directly, but is calculated at point *a* based on the $\Delta t_{a,1\to 3}$ measurement.

When there is no relative motion between *a* and *p*, and because the speed of light is constant, (C-2) makes sense because the time for the pulse to travel from *a* to *p* would be the same as the time for the reflection from p to reach a. Equation (C-3) would also make sense, because the time increment for the pulse to travel from point a-to-p-to-a would be half the time for the pulse to travel from a to p. But are (C-2) and (C-3) correct when there is relative motion between a and p?

To answer this question, consider another target at point q being stationary relative to point a. Equations (C-2) and (C-3) would then apply for target q, yielding a (C-2) distance-from-point-a solution of $x_{aq,2/a}$ at (C-3) time $t_{a,2}$. Consider now that target p has been in general motion relative to point a, but that it has reached target q at exactly the time that the e/m pulse from a arrived at q (i.e., at $t_{a,2}$). Then the distance from a to p at $t_{a,2}$ (i.e., $x_{ap,2/a}$) would be identical to $x_{aq,2/a}$. But there is no difference between this scenario and another in which point p was actually point q in motion, being at distance $x_{ap,2/a}$ when illuminated by the e/m pulse. Thus, it can be concluded that (C-2) and (C-3) will accurately measure $x_{ap,2/a}$ for any motion of point p relative to point a.

It is very fortuitous that the velocity of light is constant to any observer. This is what makes (C-2) and (C-3) accurate regardless of the relative velocity between the e/m pulse source and target. If the speed of propagation of the transmit signal was constant relative to the medium in which points a and p were immersed (e.g., a sonar pulse in water), (C-2) and (C-3) would have to account for the velocity of a and p relative to the medium, another parameter needing separate measurement.

C.2 Direction Vector To Target

The previous section discussed how e/m transmission/reflection can be used to determine 1) The distance to a target when the target was illuminated by the transmitted e/m pulse, and 2) The time instant of target illumination. At the instant of illumination (time $t_{a,2}$ on the point a clock), the reflection will generate an expanding e/m spherical wave of radiation, the distance from the wave back to the target illumination point being the same for any point on the wave, the amplitude depending on the geometry and composition of the target reflection surface. A photon in the wave will travel away from the target along a natural motion trajectory in non-rotating inertial space, deviating from a straight line only by the difference in gravity at the current photon location and at the reflection-from-target point (See Appendix A for a measurable definition of non-rotating inertial space). Thus, when the wave reaches the point a antenna, the travel direction vector for photons in the wave reaching the antenna will be along a natural motion trajectory leading back to the target point reflection event. As such, the angular direction of an inertially stabilized antenna trained toward the target will accurately measure the angular orientation of a pointing vector to the target location at time $t_{a,2}$ on the point a clock (call it $\underline{u}_{aq,2/a}$).

C.3 Distance Vector To Target

Multiplying the target pointing vector $\underline{u}_{aq,2/a}$ in C.2 by the measured target distance $x_{ap,2/a}$ in C.1 provides a distance vector measurement $\underline{x}_{ap,2/a}$ from point *a* to where target *p* was at time $t_{a,2}$ on the point *a* clock.

C.4 Time Instant Of Target Distance Vector Determination

The time instant $t_{a,2}$ associated with the $\underline{x}_{ap,2/a}$ distance vector determination in C.3 is calculated as half the time interval from e/m transmission at point *a* (at $t_{a,1}$) to the e/m reflection receive time at point a (at $t_{a,3}$) as derived previously in (C-3). The time instant $t_{a,2}$ would be the "time stamp" assigned to the $\underline{x}_{ap,2/a}$ measurement.

C.5 Relative Target Velocity Determination

The relative velocity between points *a* and *b* (e.g., $\frac{d}{dt_a}\underline{x}_{ab/a} = v_{ab}\,\underline{u}_{\upsilon}$ as defined in (8) for a point a observer) can be determined from successive e/m pulse transmission/receipt measurements as the change in $\underline{x}_{ab/a}$ over sequential point *a* clock time intervals, divided by the time interval between the successive measurements. Similarly, the relative velocity between points *b* and *a* (defined in (8) for a point *b* observer as $\frac{d}{dt_b}\underline{x}_{ba/b} = v_{ba}\,\underline{u}_{\upsilon} = -\,v_{ab}\,\underline{u}_{\upsilon}$) can be determined from successive e/m pulse transmission/receipt measurements as the change in $\underline{x}_{ba/b}$ over sequential point *b* clock time intervals, divided by the time interval between the successive measurements.

APPENDIX D - MEASURING THE DISTANCE VECTOR TO A REMOTE SPACE-TIME EVENT

This appendix provides a general example of how the distance vector to a remote event can be measured at an observation point using radar-like equipment and means to detect the event occurrence. The example assumes that the event occurrence will generate an electro-magnetic (e/m) pulse wave that travels at the speed of light from the event spatial location (e.g., a lightning strike or a pulse transmitted by a beacon at the event location), eventually arriving at the observation point where the distance vector to the event is to be determined. The method of determining the distance vector to the event is based on the speed of light being constant relative to any observer for all e/m waves, not only the e/m pulse generated by the event. Thus, if an e/m radar pulse is transmitted from the observation point (call it point a) and is reflected from the event spatial location (call it point *p*) at the event time instant, the reflected radar return pulse and the event generated pulse will arrive at point *a* at the same time instant (on a clock located at point a). Conversely then, if the radar return and the event generated pulses arrive simultaneously at point *a*, and if it is known that the radar pulse was reflected from point *p*, it can be concluded that the radar pulse traversed the same distance as the event pulse.

The previous conclusion presents a method for determining the distance vector from point *a* to spatial point *p* at the point *p* event time. As in Appendix C, define time instants 1, 2, and 3 respectively as the time a radar pulse is transmitted from point *a*, is reflected from point *p*, and is received back at point *a*. Section C.5 of Appendix C describes how the radar transmit/return time instants 1 and 3 (detected on a point *a* clock) would be used to determine the distance vector from point *a* to point *p* at the time 2 instant of radar pulse point *p* reflection (in terms of point *a* clock time) - call it $\underline{x}_{ap/a}$, the *a* to *p* distance vector as measured at point *a*. (Note that $\underline{x}_{ap/a}$ would be calculated at time instant 3 when the radar return was received, not at reflection time 2 which is calculated (using (C-3) of Appendix C) from the point *a* time instant 1 and 3 measurements.) Let us further assume that the point *a* radar has been continually tracking point *p*, hence, continually calculating $\underline{x}_{ap/a}$ measurements. Let us also assume that the event occurrence detector at point *a* (e.g., a beacon pulse receiver) is continually "listening" for the occurrence of an event at point *p*. Based on the conclusion of the previous paragraph, the distance vector from point *a* to the event at point *p* can then be determined as equal to the $\underline{x}_{ap/a}$ radar measurement at event pulse point *a* arrival time ("announced" by the event detector). The time stamp for that $\underline{x}_{ap/a}$ measurement would be the radar calculated point *p* reflection time 2 using (C-3) of Appendix C.

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