DIFFERENTIAL POINT-TO-POINT RELATIVITY IN ROTATING COORDINATES

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ABSTRACT

This article develops formulas for converting Point-To-Point Relativity kinematic equations in non-rotating coordinates into their equivalent in rotating coordinates. The formulas are then used to prepare generalized Point-To-Point Relativity equations in rotating coordinates as a function of their non-rotating equivalents.

INTRODUCTION

Point-To-Point Relativity is a revised version of traditional Special Relativity in which relative motion between observers is analytically defined by the vector distance between observation points (in contrast with traditional Relativity in which relative motion between observers is referenced to relatively translating coordinate frames). The basic Point-To-Point concept was originally described in [2] based on the classical assumption of constant velocity between observers [3 – 6]. Using [1] as a base, [2] presented a differential Point-To-Point approach that allows differential changes in relative velocity between observers. As in classical Relativity theory, [1], [2], and [3 – 6] are implicitly based on motion described in non-rotating inertial coordinates. This article expands on [2], allowing for rotation of coordinate frames in which differential Point-To-Point Relativity motion is described.

NOTATION

The following general notation is used in this article:

$\underline{V}$ = Vector parameter having length and direction. Vectors in this article are classified as "free vectors" having no preferred location in coordinate frames in which they are analytically described.

$\underline{V}^A$ = Vector $\underline{V}$ represented as a column matrix with elements equal to the projection of vector $\underline{V}$ on coordinate frame A axes. The projection of $\underline{V}$ on each frame A axis equals the dot product of $\underline{V}$ with a unit parallel to (i.e., defining) that frame A axis.

$\left(\underline{V}^A\times\right)$ = Skew symmetric (or cross-product) form of general A frame vector column $\underline{V}^A$ represented by the square matrix $\begin{bmatrix} 0 & -V_{ZA} & V_{YA} \\ V_{ZA} & 0 & -V_{XA} \\ -V_{YA} & V_{XA} & 0 \end{bmatrix}$ in which V_{XA} , V_{YA}, V_{ZA} are the components of $\underline{V}^A$. The matrix product of $\left(\underline{V}^A\times\right)$ with another A frame vector (e.g., $\underline{W}^A$) equals the A frame components of the cross-product of $\underline{V}^A$ with the other A frame vector, i.e., $\left(\underline{V}^A\times\right)\underline{W}^A = \underline{V}^A \times \underline{W}^A$.

$/i$ = Vector subscript denoting the vector parameter being observed (measured or calculated from measurements) at observation point i (i being point a or b).

Observable Event – An event at a position location in space at a particular instant in time (e.g., a lightning strike, explosion, or radar pulse illumination) that can be observed at a remote spatial location based on electro-magnetic wave propagation (e.g. light or radar) from the event to the observation point [3 pp. 29 & 36, 4 pp. 515 & 521, 5 pp. 28 & 236-238, 6 pp. 10].

DIFFERENTIAL POINT-TO-POINT RELATIVITY KINEMATIC EQUATIONS

Reference [1, Eqs. (41)] derives equations for the differential Relativistic position change of a remote point in space p observed at points a and b in motion relative to one another as

$$
\begin{aligned}
d\underline{x}_{p/a} &= d\underline{x}_{p/b} - \underline{v}_{a/b}\, dt_b + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right)\left(d\underline{x}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b} / v_{ab}^2 - \underline{v}_{a/b}\, dt_b \right) \\
dt_a &= \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(dt_b - d\underline{x}_{p/b} \cdot \underline{v}_{a/b} / c^2 \right) \\
d\underline{x}_{p/b} &= d\underline{x}_{p/a} - \underline{v}_{b/a}\, dt_a + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right)\left(d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 - \underline{v}_{b/a}\, dt_a \right) \\
dt_b &= \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(dt_a - d\underline{x}_{p/a} \cdot \underline{v}_{b/a} / c^2 \right)
\end{aligned}
\tag{1}
$$

where $d\underline{x}_{p/a}$ is the differential change in position of point p observed at point a, dt_a is the differential time increment elapsed during the $d\underline{x}_{p/a}$ movement on a clock located at point a, $\underline{v}_{b/a}$ is the instantaneous velocity of point b observed at point a, v_{ab} is the magnitude of $\underline{v}_{b/a}$, c

is the speed of light, and similarly for $d\underline{x}_{p/b}$, dt_b, $\underline{v}_{a/b}$ for observations at point *b*. Eqs. (1) are based on the speed of light being the same constant relative to any observer, a fundamental premise of Relativity theory, but also on $\underline{v}_{b/a}$ and $\underline{v}_{a/b}$ being of equal magnitude v_{ab} but oppositely directed, a fundamental premise of Newtonian [7] and Relatively theory [3 – 6]:

$$\underline{v}^B_{b/a} \equiv \frac{d\underline{x}^B_{b/a}}{dt_a} \qquad \underline{v}^B_{a/b} \equiv \frac{d\underline{x}^B_{a/b}}{dt_b} \qquad \underline{v}^B_{a/b} = -\underline{v}^B_{b/a} \qquad v_{ab} \equiv \left|\underline{v}^B_{b/a}\right| = \left|\underline{v}^B_{a/b}\right| \qquad (2)$$

The peculiar $\sqrt{1-v_{ab}^2/c^2}$ term in (1) is a unique contribution from Relativity Theory that assures that if point *p* is travelling at the speed of light, the magnitude of *p* velocity relative to observation point *a* or *b* will be the same constant *c*, i.e., from Appendix A:

$$\begin{aligned} &\left|d\underline{x}_{p/a}/dt_a\right| = \sqrt{\left(d\underline{x}_{p/a}/dt_a\right)\cdot\left(d\underline{x}_{p/a}/dt_a\right)} \\ &= \left|d\underline{x}_{p/b}/dt_b\right| = \sqrt{\left(d\underline{x}_{p/b}/dt_b\right)\cdot\left(d\underline{x}_{p/b}/dt_b\right)} = c \end{aligned} \qquad (3)$$

Eq. (3) is the reason that time intervals in (1) measured at observation points *a* and *b* (i.e., dt_a and dt_b) are unequal. In contrast, time intervals in Newtonian theory are the same at all observation points, regardless of whether there is relative velocity between observers (i.e., $dt_a = dt_b$). From (1) we see that the Newtonian condition corresponds with the magnitude of relative velocity between points *a* and *b* (i.e, $\left|\underline{v}_{b/a}\right| = \left|\underline{v}_{b/a}\right| = v_{ab}$) being negligibly small compared to *c*, i.e., $1-v_{ab}^2/c^2 \ll 1$.

Although not specifically stated in [1], the derivation of (1) is based on all observations being made in “non-rotating” inertial coordinates, the basis for both Newtonian and Relativity kinematic theory. To derive the equivalent to (1) for observations in rotating coordinates for this article, we start from the [1] basics and rebuild a revised form that accounts for coordinate frame rotation. As in [1], consider observers at points *a* and *b* observing the motion of a distant point *p*, each observer measuring the motion as the difference between observed *p* position locations (“events”) at two successive time points t_1 and t_2 (t_2 following t_1):

$$\Delta\underline{x}_{p/a} = \underline{x}_{p_2/a} - \underline{x}_{p_1/a} \qquad \Delta\underline{x}_{p/b} = \underline{x}_{p_2/b} - \underline{x}_{p_1/b} \qquad (4)$$

where $\underline{x}_{p_1/a}$, $\underline{x}_{p_2/a}$ are distance vectors (positions) measured at point *a* from point *a* to *p* at times t_1 and t_2, $\Delta\underline{x}_{p/a}$ is the change (linear translation) in the point *a* observed *p* position vector over the t_1 to t_2 time interval, and similarly for $\underline{x}_{p_1/b}$, $\underline{x}_{p_2/b}$, $\Delta\underline{x}_{p/b}$.

If points *a* and *b* have translated during the t_1 to t_2 time interval, $\Delta \underline{x}_{p/a}$ will differ from $\Delta \underline{x}_{p/b}$. Observers *a* and *b* can account for the relative translation when predicting what the other would observe:

$$\Delta \underline{x}_{bp/a} = \Delta \underline{x}_{p/a} - \Delta \underline{x}_{b/a} \qquad \Delta \underline{x}_{ap/b} = \Delta \underline{x}_{p/b} - \Delta \underline{x}_{a/b} \tag{5}$$

where $\Delta \underline{x}_{bp/a}$ is the point *a* prediction of $\Delta \underline{x}_{p/b}$ and similarly for $\Delta \underline{x}_{ap/b}$. As in (4) we can also write

$$\Delta \underline{x}_{b/a} = \underline{x}_{b_2/a} - \underline{x}_{b_1/a} \qquad \Delta \underline{x}_{a/b} = \underline{x}_{a_2/b} - \underline{x}_{a_1/b} \tag{6}$$

KINEMATIC PARAMETERS IN NON-ROTATING AND ROTATING COORDINATE FRAMES

Eqs. (4) – (6) are valid in any non-rotating coordinate frame. Let us now introduce non-rotating coordinate frames B_1 and B_2 defined as parallel to the instantaneous orientation of rotating coordinate frame *B* at successive time instants t_1 and t_2. Because B_1 and B_2 are non-rotating relative to a common non-rotating space, the angular orientation of B_2 relative to B_1 will be constant. For clarity, Eqs. (4) and (6) are now rewritten in B_1 and B_2 coordinates as

$$\begin{aligned} \Delta \underline{x}^{B_1}_{p/a} &= \underline{x}^{B_1}_{p_2/a} - \underline{x}^{B_1}_{p_1/a} \qquad \Delta \underline{x}^{B_1}_{p/b} = \underline{x}^{B_1}_{p_2/b} - \underline{x}^{B_1}_{p_1/b} \\ \Delta \underline{x}^{B_1}_{b/a} &= \underline{x}^{B_1}_{b_2/a} - \underline{x}^{B_1}_{b_1/a} \qquad \Delta \underline{x}^{B_1}_{a/b} = \underline{x}^{B_1}_{a_2/b} - \underline{x}^{B_1}_{a_1/b} \end{aligned} \tag{7}$$

$$\begin{aligned} \Delta \underline{x}^{B_2}_{p/a} &= \underline{x}^{B_2}_{p_2/a} - \underline{x}^{B_2}_{p_1/a} \qquad \Delta \underline{x}^{B_2}_{p/b} = \underline{x}^{B_2}_{p_2/b} - \underline{x}^{B_2}_{p_1/b} \\ \Delta \underline{x}^{B_2}_{b/a} &= \underline{x}^{B_2}_{b_2/a} - \underline{x}^{B_2}_{b_1/a} \qquad \Delta \underline{x}^{B_2}_{a/b} = \underline{x}^{B_2}_{a_2/b} - \underline{x}^{B_2}_{a_1/b} \end{aligned} \tag{8}$$

where the B_1 and B_2 superscript identifies the coordinate frame on which the vector components are projected (i.e., as elements of a column matrix).

The components of (7) – (8) observed at point *a* are related through

$$\begin{aligned} \Delta \underline{x}^{B_1}_{p/a} &= C^{B_1}_{B_2} \underline{x}^{B_2}_{p_2/a} - \underline{x}^{B_1}_{p_1/a} = \left(C^{B_1}_{B_2} - I + I \right) \underline{x}^{B_2}_{p_2/a} - \underline{x}^{B_1}_{p_1/a} = \underline{x}^{B_2}_{p_2/a} - \underline{x}^{B_1}_{p_1/a} + \left(C^{B_1}_{B_2} - I \right) \underline{x}^{B_2}_{p_2/a} \\ \Delta \underline{x}^{B_1}_{b/a} &= C^{B_1}_{B_2} \underline{x}^{B_2}_{b_2/a} - \underline{x}^{B_1}_{b_1/a} = \left(C^{B_1}_{B_2} - I + I \right) \underline{x}^{B_2}_{b_2/a} - \underline{x}^{B_1}_{b_1/a} = \underline{x}^{B_2}_{b_2/a} - \underline{x}^{B_1}_{b_1/a} + \left(C^{B_1}_{B_2} - I \right) \underline{x}^{B_2}_{b_2/a} \end{aligned} \tag{9}$$

where I is the identity matrix and $C_{B_2}^{B_1}$ is a direction cosine matrix that transforms vector components from their values in inertial coordinate frame B_2 to their values in inertial coordinate frame B_1.

Since non-rotating frames B_1 and B_2 are defined as aligned with the rotating B frame at time instants 1 and 2, we can calculate the change in B frame values of a vector over time instants 1 and 2 as the difference between the B_1 and B_2 projected values. Thus, the equivalent to (7) and (8) in the rotating B frame is:

$$\Delta\underline{\chi}_{p/a}^{B} \equiv \underline{x}_{p_2/a}^{B_2} - \underline{x}_{p_1/a}^{B_1} \qquad \Delta\underline{\chi}_{b/a}^{B} \equiv \underline{x}_{b_2/a}^{B_2} - \underline{x}_{b_1/a}^{B_1} \tag{10}$$

where $\Delta\underline{\chi}_{p/a}^{B}$, $\Delta\underline{\chi}_{b/a}^{B}$ are the changes in distance vectors from observation point a to points p and b as measured at point a in the rotating B frame (superscript) over the t_1 to t_2 time interval. We also note that $C_{B_2}^{B_1} - I$ in (9) represents the change in the $C_B^{B_1}$ direction cosine matrix from its identity value at t_1 (when $B = B_1$) to its $C_{B_2}^{B_1}$ value at t_2. For the angular rotation over t_1 to t_2, $C_{B_2}^{B_1} - I$ can be equated to the equivalent rotation angle vector $\Delta\underline{\theta}_{IB}^{B}$ which for small angular rotation approximates as [8, Sect. 3.5.2]:

$$C_{B_2}^{B_1} - I \approx \left(\Delta\underline{\theta}_{IB}^{B} \times\right) \tag{11}$$

where the IB subscript indicates the angular rotation of frame B from inertially non-rotating frame B_1 to inertially non-rotating frame B_2. Substituting (10) and (11) in (9) then obtains

$$\Delta\underline{x}_{p/a}^{B_1} = \Delta\underline{\chi}_{p/a}^{B} + \Delta\underline{\theta}_{IB}^{B} \times \underline{x}_{p_2/a}^{B_2} \qquad \Delta\underline{x}_{b/a}^{B_1} = \Delta\underline{\chi}_{b/a}^{B} + \Delta\underline{\theta}_{IB}^{B} \times \underline{x}_{b_2/a}^{B_2} \tag{12}$$

Finally, we let the Δ changes be infinitesimally small so that $\underline{x}_{p_2/a}^{B_2} \rightarrow \underline{x}_{p/a}^{B}$, $\underline{x}_{b_2/a}^{B_2} \rightarrow \underline{x}_{b/a}^{B}$, $\Delta\underline{x}_{p/a}^{B_1} \rightarrow d\,\underline{x}_{p/a}^{B}$, $\Delta\underline{\chi}_{p/a}^{B} \rightarrow d\,\underline{\chi}_{p/a}^{B}$, $\Delta\underline{x}_{b/a}^{B_1} \rightarrow d\,\underline{x}_{b/a}^{B}$, $\Delta\underline{\chi}_{b/a}^{B} \rightarrow d\,\underline{\chi}_{b/a}^{B}$, and $\Delta\underline{\theta}_{IB}^{B} \rightarrow d\,\underline{\theta}_{IB}^{B}$. Then (12) becomes

$$d\,\underline{x}_{p/a}^{B} = d\,\underline{\chi}_{p/a}^{B} + d\,\underline{\theta}_{IB}^{B} \times \underline{x}_{p/a}^{B} \qquad d\,\underline{x}_{b/a}^{B} = d\,\underline{\chi}_{b/a}^{B} + d\,\underline{\theta}_{IB}^{B} \times \underline{x}_{b/a}^{B} \tag{13}$$

The same process yields for the point b observed position changes

$$d\,\underline{x}_{p/b}^{B} = d\,\underline{\chi}_{p/b}^{B} + d\,\underline{\theta}_{IB}^{B} \times \underline{x}_{p/b}^{B} \qquad d\,\underline{x}_{a/b}^{B} = d\,\underline{\chi}_{a/b}^{B} + d\,\underline{\theta}_{IB}^{B} \times \underline{x}_{a/b}^{B} \tag{14}$$

Similar to (2), we can also define the relative linear velocity between observation points *a* and *b*,

$$\underline{V}_{b/a}^{B} \equiv \frac{d\underline{\chi}_{b/a}^{B}}{dt_a} \qquad \underline{V}_{a/b}^{B} \equiv \frac{d\underline{\chi}_{a/b}^{B}}{dt_b} \tag{15}$$

where in rotating *B* frame coordinates, $\underline{V}_{b/a}^{B}$ is the instantaneous velocity of point *b* observed at point *a*, and similarly for $\underline{V}_{a/b}^{B}$ at point *b*. Dividing the $d\underline{x}_{b/a}^{B}$ and $d\underline{x}_{a/b}^{B}$ expressions in (13) and (14) by the corresponding time interval and applying (15) obtains

$$\frac{d\underline{x}_{b/a}^{B}}{dt_a} = \underline{V}_{b/a}^{B} + d\underline{\theta}_{IB}^{B} \times \underline{x}_{b/a}^{B} / dt_a \qquad \frac{d\underline{x}_{a/b}^{B}}{dt_b} = \underline{V}_{a/b}^{B} + d\underline{\theta}_{IB}^{B} \times \underline{x}_{a/b}^{B} / dt_b \tag{16}$$

Applying the (2) velocity definitions for $\frac{d\underline{x}_{b/a}^{B}}{dt_a}$ and $\frac{d\underline{x}_{a/b}^{B}}{dt_b}$, (16) becomes

$$\frac{d\underline{x}_{b/a}^{B}}{dt_a} = \underline{v}_{b/a}^{B} = \underline{V}_{b/a}^{B} + d\underline{\theta}_{IB}^{B} \times \underline{x}_{b/a}^{B} / dt_a \qquad \frac{d\underline{x}_{a/b}^{B}}{dt_b} = -\underline{v}_{b/a}^{B} = \underline{V}_{a/b}^{B} + d\underline{\theta}_{IB}^{B} \times \underline{x}_{a/b}^{B} / dt_b \tag{17}$$

where $\underline{v}_{b/a}^{B}$ is now more specifically defined as the instantaneous velocity of point *b* relative to point *a* as measured in a <u>non-rotating</u> coordinate frame that is instantaneously aligned with the <u>rotating</u> *B* frame.

The first expression in (17) also shows with (2) that

$$\begin{aligned} v_{ab}^{2} &= \underline{v}_{b/a}^{B} \cdot \underline{v}_{b/a}^{B} = \left(\underline{V}_{b/a}^{B} + d\underline{\theta}_{IB}^{B} \times \underline{x}_{b/a}^{B} / dt_a\right) \cdot \left(\underline{V}_{b/a}^{B} + d\underline{\theta}_{IB}^{B} \times \underline{x}_{b/a}^{B} / dt_a\right) \\ &= V_{b/a}^{2} + 2\,\underline{V}_{b/a}^{B} \cdot \left(d\underline{\theta}_{IB}^{B} \times \underline{x}_{b/a}^{B}\right) / dt_a + \left(d\underline{\theta}_{IB}^{B} \times \underline{x}_{b/a}^{B}\right) \cdot \left(d\underline{\theta}_{IB}^{B} \times \underline{x}_{b/a}^{B}\right) / \left(dt_a\right)^{2} \end{aligned} \tag{18}$$

where $V_{b/a}$ is the magnitude of $\underline{V}_{b/a}$ in any *B* frame rotating through $d\underline{\theta}_{IB}^{B}$ relative to non-rotating inertial space. Similarly, the second expression in (17) shows that

$$\begin{aligned} v_{ab}^{2} &= \underline{v}_{a/b}^{B} \cdot \underline{v}_{a/b}^{B} = \left(\underline{V}_{a/b}^{B} + d\underline{\theta}_{IB}^{B} \times \underline{x}_{a/b}^{B} / dt_b\right) \cdot \left(\underline{V}_{a/b}^{B} + d\underline{\theta}_{IB}^{B} \times \underline{x}_{a/b}^{B} / dt_b\right) \\ &= V_{a/b}^{2} + 2\,\underline{V}_{a/b}^{B} \cdot \left(d\underline{\theta}_{IB}^{B} \times \underline{x}_{a/b}^{B}\right) / dt_b + \left(d\underline{\theta}_{IB}^{B} \times \underline{x}_{a/b}^{B}\right) \cdot \left(d\underline{\theta}_{IB}^{B} \times \underline{x}_{a/b}^{B}\right) / \left(dt_b\right)^{2} \end{aligned} \tag{19}$$

The pertinent results from (2), (13), (14), (15), (17), (18), and (19) summarize as follows

$$\underline{v}_{b/a}^{B} \equiv \frac{d\,\underline{x}_{b/a}^{B}}{dt_a} \qquad \underline{v}_{a/b}^{B} \equiv \frac{d\,\underline{x}_{a/b}^{B}}{dt_b} \qquad \underline{v}_{a/b}^{B} = -\,\underline{v}_{b/a}^{B}$$

$$\underline{V}_{b/a}^{B} \equiv \frac{d\,\underline{\chi}_{b/a}^{B}}{dt_a} \qquad \underline{V}_{a/b}^{B} \equiv \frac{d\,\underline{\chi}_{a/b}^{B}}{dt_b}$$

$$d\,\underline{x}_{p/a}^{B} = d\,\underline{\chi}_{p/a}^{B} + d\,\underline{\theta}_{IB}^{B} \times \underline{x}_{p/a}^{B} \qquad d\,\underline{x}_{p/b}^{B} = d\,\underline{\chi}_{p/b}^{B} + d\,\underline{\theta}_{IB}^{B} \times \underline{x}_{p/b}^{B} \tag{20}$$

$$\underline{v}_{b/a}^{B} = \underline{V}_{b/a}^{B} + d\,\underline{\theta}_{IB}^{B} \times \underline{x}_{b/a}^{B} \,/\, dt_a \qquad \underline{v}_{a/b}^{B} = \underline{V}_{a/b}^{B} + d\,\underline{\theta}_{IB}^{B} \times \underline{x}_{a/b}^{B} \,/\, dt_b$$

$$v_{ab}^{2} = V_{b/a}^{2} + 2\,\underline{V}_{b/a}^{B} \cdot \left(d\,\underline{\theta}_{IB}^{B} \times \underline{x}_{b/a}^{B}\right) / \,dt_a + \left(d\,\underline{\theta}_{IB}^{B} \times \underline{x}_{b/a}^{B}\right) \cdot \left(d\,\underline{\theta}_{IB}^{B} \times \underline{x}_{b/a}^{B}\right) / \left(dt_a\right)^2$$

$$= V_{a/b}^{2} + 2\,\underline{V}_{a/b}^{B} \cdot \left(d\,\underline{\theta}_{IB}^{B} \times \underline{x}_{a/b}^{B}\right) / \,dt_b + \left(d\,\underline{\theta}_{IB}^{B} \times \underline{x}_{a/b}^{B}\right) \cdot \left(d\,\underline{\theta}_{IB}^{B} \times \underline{x}_{a/b}^{B}\right) / \left(dt_b\right)^2$$

Equations (20) can now be substituted into (1) to find differential point-to-point Relativity conversion equations in rotating coordinates.

CONVERTING TO AND FROM ROTATING COORDINATES

Three types of conversion formulas can be defined, 1) Converting rotating coordinate frame effects into non-rotating coordinates, 2) Converting rotating coordinate frame effects viewed by observer *a* with their equivalent rotating coordinate frame effects viewed by observer *b* (and the converse for rotating observer *b* effects viewed by rotating observer *a*), and 3) Converting non-rotating coordinate frame effects into rotating coordinates. The 3) conversion equations are the simplest and will be presented last. The 1) and 2) conversion equations are complex, but analytically straight-forward in their derivation. The method of deriving the 1) and 2) formulas will only be outlined.

To convert rotating coordinate frame effects into non-rotating coordinates, maintain the left side of (1) as shown, and substitute the following expressions from (20) on the right side: $d\,\underline{x}_{p/a}^{B}$, $d\,\underline{x}_{p/b}^{B}$, $\underline{v}_{b/a}^{B}$, $\underline{v}_{a/b}^{B}$, v_{ab}^{2}. To convert rotating coordinate frame effects observed by one observer into rotating coordinates observed by the other observer, substitute the previous expressions from (20) in both the left and right side of (1).

To convert non-rotating coordinate frame effects into rotating coordinates, maintain the right side of (1) as shown, and substitute the $d\,\underline{x}_{p/a}^{B}$ and $d\,\underline{x}_{p/b}^{B}$ expressions from (20) on the left. Having the $d\,\underline{x}_{p/a}^{B}$ and $d\,\underline{x}_{p/b}^{B}$ conversion formulas in (20) defined in the *B* frame, and the definition for rotating *B* frame coordinates previously defined as being instantaneously aligned with a non-rotating *B* frame, the *B* frame subscript notation in the conversion formulas can be eliminated yielding for the final result:

$$
\begin{gathered}
d\underline{\chi}_{p/a} + d\underline{\theta} \times \underline{x}_{p/a} \\
= d\underline{x}_{p/b} - \underline{v}_{a/b}\, dt_b + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(d\underline{x}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b} / v_{ab}^2 - \underline{v}_{a/b}\, dt_b \right) \\
dt_a = \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(dt_b - d\underline{x}_{p/b} \cdot \underline{v}_{a/b} / c^2 \right) \\
d\underline{\chi}_{p/b} + d\underline{\theta} \times \underline{x}_{p/b} \\
= d\underline{x}_{p/a} - \underline{v}_{b/a}\, dt_a + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 - \underline{v}_{b/a}\, dt_a \right) \\
dt_b = \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(dt_a - d\underline{x}_{p/a} \cdot \underline{v}_{b/a} / c^2 \right)
\end{gathered} \tag{21}
$$

In (21), $d\underline{\theta}$ vector represents that incremental angular rotation of the rotating coordinate frame relative to non-rotating inertial space.

APPENDIX A

Demonstrating With Point-To-Point Relativity Kinematics That Observers Travelling Relative To Each Other Will Measure The Same Speed For An Observed Remote Point Travelling At The Speed Of Light

This appendix demonstrates with Point-To-Point Relativity Eqs. (1) that when a point *p* is travelling at a speed of light velocity as observed at point *a*, the point *p* velocity observed at another point *b* in motion relative to point *a*, will also be at the speed of light. The derivation begins by first defining

$$
\underline{v}_{p/b} \equiv \frac{d\underline{x}_{p/b}}{dt_b} \qquad \underline{v}_{p/a} \equiv \frac{d\underline{x}_{p/a}}{dt_a} \tag{A-1}
$$

where $\underline{v}_{p/a}$ and $\underline{v}_{p/b}$ are velocities of point *p* determined at observation points *a* and *b*. Dividing the $d\underline{x}_{p/b}$ and dt_b equations in (1) by dt_a obtains with (A-1) and $\underline{v}_{b/a} = v_{ab}\, \underline{u}_v$ from (2):

$$\frac{d\underline{x}_{p/b}}{dt_a} = \underline{v}_{p/b}\frac{dt_b}{dt_a} = \underline{v}_{p/a} - v_{ab}\,\underline{u}_v + \left(\frac{1}{\sqrt{1-v_{ab}^2/c^2}} - 1\right)\left(\underline{v}_{p/a}\cdot\underline{u}_v\,\underline{u}_v - v_{ab}\,\underline{u}_v\right)$$

$$= \underline{v}_{p/a} - \underline{v}_{p/a}\cdot\underline{u}_v\,\underline{u}_v + \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(\underline{v}_{p/a}\cdot\underline{u}_v - v_{ab}\right)\underline{u}_v \qquad \text{(A-2)}$$

$$\frac{dt_b}{dt_a} = \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(1 - \underline{v}_{p/a}\cdot\underline{u}_v\,v_{ab}/c^2\right)$$

The $\underline{v}_{p/a}$ velocity in (A-3) is then defined as having light speed c so that

$$\underline{v}_{p/a} = c\,\underline{u}_{p/a} \qquad \text{(A-3)}$$

where $\underline{u}_{p/a}$ is a unit vector in the direction of $\underline{v}_{p/a}$. Substituting (A-3) in (A-2) obtains

$$\underline{v}_{p/b}\frac{dt_b}{dt_a} = c\left[\underline{u}_{p/a} - \underline{u}_{p/a}\cdot\underline{u}_v\,\underline{u}_v + \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(\underline{u}_{p/a}\cdot\underline{u}_v - v_{ab}/c\right)\underline{u}_v\right] \qquad \text{(A-4)}$$

$$\frac{dt_b}{dt_a} = \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(1 - \underline{u}_{p/a}\cdot\underline{u}_v\,v_{ab}/c\right)$$

Recognizing that $\underline{u}_{p/a} - \underline{u}_{p/a}\cdot\underline{u}_v\,\underline{u}_v$ in (A-4) is perpendicular to $\underline{u}_v$ allows application of the Pythagorian Theorem to obtain for the magnitude squared of $\underline{v}_{p/b}\,dt_b/dt_a$:

$$\underline{v}_{p/b}\cdot\underline{v}_{p/b}\left(\frac{dt_b}{dt_a}\right)^2/c^2 = \frac{\left(\underline{u}_{p/a}\cdot\underline{u}_v - v_{ab}/c\right)^2}{\left(1-v_{ab}^2/c^2\right)} + \left(\underline{u}_{p/a} - \underline{u}_{p/a}\cdot\underline{u}_v\,\underline{u}_v\right)\cdot\left(\underline{u}_{p/a} - \underline{u}_{p/a}\cdot\underline{u}_v\,\underline{u}_v\right)$$

$$= \frac{\left(\underline{u}_{p/a}\cdot\underline{u}_v\right)^2 - 2\,\underline{u}_{p/a}\cdot\underline{u}_v\,v_{ab}/c + v_{ab}^2/c^2}{\left(1-v_{ab}^2/c^2\right)} + 1 - \left(\underline{u}_{p/a}\cdot\underline{u}_v\right)^2 \qquad \text{(A-5)}$$

$$= \frac{\left[\begin{array}{c}\left(\underline{u}_{p/a}\cdot\underline{u}_v\right)^2 - 2\,\underline{u}_{p/a}\cdot\underline{u}_v\,v_{ab}/c + v_{ab}^2/c^2 \\ +1 - \left(\underline{u}_{p/a}\cdot\underline{u}_v\right)^2 - v_{ab}^2/c^2 + \left(\underline{u}_{p/a}\cdot\underline{u}_v\right)^2 v_{ab}^2/c^2\end{array}\right]}{\left(1-v_{ab}^2/c^2\right)}$$

(Continued)

$$= \frac{-2\,\underline{u}_{p/a}\cdot\underline{u}_v\,v_{ab}/c+1+\left(\underline{u}_{p/a}\cdot\underline{u}_v\right)^2 v_{ab}^2/c^2}{\left(1-v_{ab}^2/c^2\right)} = \frac{\left(1-\underline{u}_{p/a}\cdot\underline{u}_v\,v_{ab}/c\right)^2}{\left(1-v_{ab}^2/c^2\right)} \qquad \text{(A-5) Concluded}$$

Identifying the (A-5) result as the square of the dt_b / dt_a term in (A-4) then shows that

$$\underline{v}_{p/b}\cdot\underline{v}_{p/b} = c^2 \qquad \text{(A-6)}$$

Thus we see from (A-6), that when the magnitude of $\underline{v}_{p/a}$ in (A-3) is the speed of light, the magnitude of $\underline{v}_{p/b}$ will also be the speed of light:

$$\left|\underline{v}_{p/b}\right| = \sqrt{\underline{v}_{p/b}\cdot\underline{v}_{p/b}} = c \qquad \text{(A-7)}$$

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