DIFFERENTIAL POINT-TO-POINT RELATIVITY IN ROTATING COORDINATES

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ABSTRACT

This article develops formulas for converting Point-To-Point Relativity kinematic equations in non-rotating coordinates into their equivalent in rotating coordinates. The formulas are then used to prepare generalized Point-To-Point Relativity equations in rotating coordinates as a function of their non-rotating equivalents. An example is provided showing how the general result can be specialized to describe the operation of fiber optic and ring laser optical gyros.

INTRODUCTION

Point-To-Point Relativity is a revised version of traditional Special Relativity in which relative motion between observers is analytically defined by the vector distance between observation points (in contrast with traditional Relativity in which relative motion between observers is referenced to relatively translating coordinate frames). The basic Point-To-Point concept was originally described in [2] based on the classical assumption of constant velocity between observers [3 – 6]. Using [1] as a base, [2] presented a differential Point-To-Point approach that allows differential changes in relative velocity between observers. As in classical Relativity theory, [1], [2], and [3 – 6] are implicitly based on motion described in non-rotating inertial coordinates. This article expands on [2], allowing for rotation of coordinate frames in which differential Point-To-Point Relativity motion is described. An example is described at the end of the article showing how the rotating frame equations can be specialized to describe the operation of optical gyros.

NOTATION

The following general notation is used in this article:

$\underline{V}$ = Vector parameter having length and direction. Vectors in this article are classified as "free vectors" having no preferred location in coordinate frames in which they are analytically described.

$\underline{V}^A$ = Vector $\underline{V}$ represented as a column matrix with elements equal to the projection of vector $\underline{V}$ on coordinate frame A axes. The projection of $\underline{V}$ on each frame A axis equals the dot product of $\underline{V}$ with a unit parallel to (i.e., defining) that frame A axis.

$\left(\underline{V}^A \times\right)$ = Skew symmetric (or cross-product) form of general A frame vector column $\underline{V}^A$ represented by the square matrix $\begin{bmatrix} 0 & -V_{ZA} & V_{YA} \\ V_{ZA} & 0 & -V_{XA} \\ -V_{YA} & V_{XA} & 0 \end{bmatrix}$ in which V_{XA}, V_{YA}, V_{ZA} are the components of $\underline{V}^A$. The matrix product of $\left(\underline{V}^A \times\right)$ with another A frame vector (e.g., $\underline{W}^A$) equals the A frame components of the cross-product of $\underline{V}^A$ with the other A frame vector, i.e., $\left(\underline{V}^A \times\right)\underline{W}^A = \underline{V}^A \times \underline{W}^A$.

$/i$ = Vector subscript denoting the vector parameter being observed (measured or calculated from measurements) at observation point i (i being point a or b).

Observable Event – An event at a position location in space at a particular instant in time (e.g., a lightning strike, explosion, or radar pulse illumination) that can be observed at a remote spatial location based on electro-magnetic wave propagation (e.g. light or radar) from the event to the observation point [3 pp. 29 & 36, 4 pp. 515 & 521, 5 pp. 28 & 236-238, 6 pp. 10].

DIFFERENTIAL POINT-TO-POINT RELATIVITY KINEMATIC EQUATIONS

Reference [1, Eqs. (41)] derives equations for the differential Relativistic position change of a remote point in space p observed at points a and b in motion relative to one another as

$$\begin{gathered}
d\underline{x}_{p/a} = d\underline{x}_{p/b} - \underline{v}_{a/b}\, dt_b + \left(\frac{1}{\sqrt{1-v_{ab}^2/c^2}} - 1\right)\left(d\underline{x}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b} / v_{ab}^2 - \underline{v}_{a/b}\, dt_b\right) \\
dt_a = \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(dt_b - d\underline{x}_{p/b} \cdot \underline{v}_{a/b} / c^2\right) \\
d\underline{x}_{p/b} = d\underline{x}_{p/a} - \underline{v}_{b/a}\, dt_a + \left(\frac{1}{\sqrt{1-v_{ab}^2/c^2}} - 1\right)\left(d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 - \underline{v}_{b/a}\, dt_a\right) \\
dt_b = \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(dt_a - d\underline{x}_{p/a} \cdot \underline{v}_{b/a} / c^2\right)
\end{gathered} \tag{1}$$

where $d\underline{x}_{p/a}$ is the differential change in position of point *p* observed at point *a*, dt_a is the differential time increment elapsed during the $d\underline{x}_{p/a}$ movement on a clock located at point *a*, $\underline{v}_{b/a}$ is the instantaneous velocity of point *b* observed at point *a*, v_{ab} is the magnitude of $\underline{v}_{b/a}$, *c* is the speed of light, and similarly for $d\underline{x}_{p/b}$, dt_b, $\underline{v}_{a/b}$ for observations at point *b*. Eqs. (1) are based on the speed of light being the same constant relative to any observer, a fundamental premise of Relativity theory, but also on $\underline{v}_{b/a}$ and $\underline{v}_{a/b}$ being of equal magnitude v_{ab} but oppositely directed, a fundamental premise of Newtonian [7] and Relatively theory [3 – 6]:

$$\underline{v}^B_{b/a} \equiv \frac{d\underline{x}^B_{b/a}}{dt_a} \qquad \underline{v}^B_{a/b} \equiv \frac{d\underline{x}^B_{a/b}}{dt_b} \qquad \underline{v}^B_{a/b} = -\underline{v}^B_{b/a} \qquad v_{ab} \equiv \left|\underline{v}^B_{b/a}\right| = \left|\underline{v}^B_{a/b}\right| \tag{2}$$

The peculiar $\sqrt{1 - v_{ab}^2 / c^2}$ term in (1) is a unique contribution from Relativity Theory that assures that if point *p* is travelling at the speed of light, the magnitude of *p* velocity relative to observation point *a* or *b* will be the same constant *c*, i.e., from Appendix A:

$$\begin{aligned} &\left| d\underline{x}_{p/a} / dt_a \right| = \sqrt{\left(d\underline{x}_{p/a} / dt_a\right) \cdot \left(d\underline{x}_{p/a} / dt_a\right)} \\ &= \left| d\underline{x}_{p/b} / dt_b \right| = \sqrt{\left(d\underline{x}_{p/b} / dt_b\right) \cdot \left(d\underline{x}_{p/b} / dt_b\right)} = c \end{aligned} \tag{3}$$

Eq. (3) is the reason that time intervals in (1) measured at observation points *a* and *b* (i.e., dt_a and dt_b) are unequal. In contrast, time intervals in Newtonian theory are the same at all observation points, regardless of whether there is relative velocity between observers (i.e., $dt_a = dt_b$). From (1) we see that the Newtonian condition corresponds with the magnitude of relative velocity between points *a* and *b* (i.e, $\left|\underline{v}_{b/a}\right| = \left|\underline{v}_{b/a}\right| = v_{ab}$) being negligibly small compared to *c*, i.e., $1 - v_{ab}^2 / c^2 \ll 1$.

Although not specifically stated in [1], the derivation of (1) is based on all observations being made in "non-rotating" inertial coordinates, the basis for both Newtonian and Relativity kinematic theory. To derive the equivalent to (1) for observations in rotating coordinates for this article, we start from the [1] basics and rebuild a revised form that accounts for coordinate frame rotation. As in [1], consider observers at points *a* and *b* observing the motion of a distant point *p*, each observer measuring the motion as the difference between observed *p* position locations ("events") at two successive time points t_1 and t_2 (t_2 following t_1):

$$\Delta\underline{x}_{p/a} = \underline{x}_{p_2/a} - \underline{x}_{p_1/a} \qquad \Delta\underline{x}_{p/b} = \underline{x}_{p_2/b} - \underline{x}_{p_1/b} \tag{4}$$

where $\underline{x}_{p_1/a}$, $\underline{x}_{p_2/a}$ are distance vectors (positions) measured at point a from point a to p at times t_1 and t_2, $\Delta\underline{x}_{p/a}$ is the change (linear translation) in the point a observed p position vector over the t_1 to t_2 time interval, and similarly for $\underline{x}_{p_1/b}$, $\underline{x}_{p_2/b}$, $\Delta\underline{x}_{p/b}$.

If points a and b have translated during the t_1 to t_2 time interval, $\Delta\underline{x}_{p/a}$ will differ from $\Delta\underline{x}_{p/b}$. Observers a and b can account for the relative translation when predicting what the other would observe:

$$\Delta\underline{x}_{bp/a} = \Delta\underline{x}_{p/a} - \Delta\underline{x}_{b/a} \qquad \Delta\underline{x}_{ap/b} = \Delta\underline{x}_{p/b} - \Delta\underline{x}_{a/b} \tag{5}$$

where $\Delta\underline{x}_{bp/a}$ is the point a prediction of $\Delta\underline{x}_{p/b}$ and similarly for $\Delta\underline{x}_{ap/b}$. As in (4) we can also write

$$\Delta\underline{x}_{b/a} = \underline{x}_{b_2/a} - \underline{x}_{b_1/a} \qquad \Delta\underline{x}_{a/b} = \underline{x}_{a_2/b} - \underline{x}_{a_1/b} \tag{6}$$

KINEMATIC PARAMETERS IN NON-ROTATING AND ROTATING COORDINATE FRAMES

Eqs. (4) – (6) are valid in any non-rotating coordinate frame. Let us now introduce non-rotating coordinate frames B_1 and B_2 defined as parallel to the instantaneous orientation of rotating coordinate frame B at successive time instants t_1 and t_2. Because B_1 and B_2 are non-rotating relative to a common non-rotating space, the angular orientation of B_2 relative to B_1 will be constant. For clarity, Eqs. (4) and (6) are now rewritten in B_1 and B_2 coordinates as

$$\begin{aligned} \Delta\underline{x}_{p/a}^{B_1} &= \underline{x}_{p_2/a}^{B_1} - \underline{x}_{p_1/a}^{B_1} \qquad \Delta\underline{x}_{p/b}^{B_1} = \underline{x}_{p_2/b}^{B_1} - \underline{x}_{p_1/b}^{B_1} \\ \Delta\underline{x}_{b/a}^{B_1} &= \underline{x}_{b_2/a}^{B_1} - \underline{x}_{b_1/a}^{B_1} \qquad \Delta\underline{x}_{a/b}^{B_1} = \underline{x}_{a_2/b}^{B_1} - \underline{x}_{a_1/b}^{B_1} \end{aligned} \tag{7}$$

$$\begin{aligned} \Delta\underline{x}_{p/a}^{B_2} &= \underline{x}_{p_2/a}^{B_2} - \underline{x}_{p_1/a}^{B_2} \qquad \Delta\underline{x}_{p/b}^{B_2} = \underline{x}_{p_2/b}^{B_2} - \underline{x}_{p_1/b}^{B_2} \\ \Delta\underline{x}_{b/a}^{B_2} &= \underline{x}_{b_2/a}^{B_2} - \underline{x}_{b_1/a}^{B_2} \qquad \Delta\underline{x}_{a/b}^{B_2} = \underline{x}_{a_2/b}^{B_2} - \underline{x}_{a_1/b}^{B_2} \end{aligned} \tag{8}$$

where the B_1 and B_2 superscript identifies the coordinate frame on which the vector components are projected (i.e., as elements of a column matrix).

The components of (7) – (8) observed at point a are related through

$$\Delta \underline{x}_{p/a}^{B_1} = C_{B_2}^{B_1} \underline{x}_{p_2/a}^{B_2} - \underline{x}_{p_1/a}^{B_1} = \left(C_{B_2}^{B_1} - I + I \right) \underline{x}_{p_2/a}^{B_2} - \underline{x}_{p_1/a}^{B_1} = \underline{x}_{p_2/a}^{B_2} - \underline{x}_{p_1/a}^{B_1} + \left(C_{B_2}^{B_1} - I \right) \underline{x}_{p_2/a}^{B_2}$$
$$\Delta \underline{x}_{b/a}^{B_1} = C_{B_2}^{B_1} \underline{x}_{b_2/a}^{B_2} - \underline{x}_{b_1/a}^{B_1} = \left(C_{B_2}^{B_1} - I + I \right) \underline{x}_{b_2/a}^{B_2} - \underline{x}_{b_1/a}^{B_1} = \underline{x}_{b_2/a}^{B_2} - \underline{x}_{b_1/a}^{B_1} + \left(C_{B_2}^{B_1} - I \right) \underline{x}_{b_2/a}^{B_2} \tag{9}$$

where I is the identity matrix and $C_{B_2}^{B_1}$ is a direction cosine matrix that transforms vector components from their values in inertial coordinate frame B_2 to their values in inertial coordinate frame B_1.

Since non-rotating frames B_1 and B_2 are defined as aligned with the rotating B frame at time instants 1 and 2, we can calculate the change in B frame values of a vector over time instants 1 and 2 as the difference between the B_1 and B_2 projected values. Thus, the equivalent to (7) and (8) in the rotating B frame is:

$$\Delta \underline{\chi}_{p/a}^{B} \equiv \underline{x}_{p_2/a}^{B_2} - \underline{x}_{p_1/a}^{B_1} \qquad \Delta \underline{\chi}_{b/a}^{B} \equiv \underline{x}_{b_2/a}^{B_1} - \underline{x}_{b_1/a}^{B_1} \tag{10}$$

where $\Delta \underline{\chi}_{p/a}^{B}$, $\Delta \underline{\chi}_{b/a}^{B}$ are the changes in distance vectors from observation point a to points p and b as measured at point a in the rotating B frame (superscript) over the t_1 to t_2 time interval. We also note that $C_{B_2}^{B_1} - I$ in (9) represents the change in the $C_{B}^{B_1}$ direction cosine matrix from its identity value at t_1 (when $B = B_1$) to its $C_{B_2}^{B_1}$ value at t_2. For the angular rotation over t_1 to t_2, $C_{B_2}^{B_1} - I$ can be equated to the equivalent rotation angle vector $\Delta \underline{\theta}_{IB}^{B}$ which for small angular rotation approximates as [8, Sect. 3.5.2]:

$$C_{B_2}^{B_1} - I \approx \left(\Delta \underline{\theta}_{IB}^{B} \times \right) \tag{11}$$

where the IB subscript indicates the angular rotation of frame B from inertially non-rotating frame B_1 to inertially non-rotating frame B_2. Substituting (10) and (11) in (9) then obtains

$$\Delta \underline{x}_{p/a}^{B_1} = \Delta \underline{\chi}_{p/a}^{B} + \Delta \underline{\theta}_{IB}^{B} \times \underline{x}_{p_2/a}^{B_2} \qquad \Delta \underline{x}_{b/a}^{B_1} = \Delta \underline{\chi}_{b/a}^{B} + \Delta \underline{\theta}_{IB}^{B} \times \underline{x}_{b_2/a}^{B_2} \tag{12}$$

Finally, we let the Δ changes be infinitesimally small so that $\underline{x}_{p_2/a}^{B_2} \rightarrow \underline{x}_{p/a}^{B}$, $\underline{x}_{b_2/a}^{B_2} \rightarrow \underline{x}_{b/a}^{B}$, $\Delta \underline{x}_{p/a}^{B_1} \rightarrow d\underline{x}_{p/a}^{B}$, $\Delta \underline{\chi}_{p/a}^{B} \rightarrow d\underline{\chi}_{p/a}^{B}$, $\Delta \underline{x}_{b/a}^{B_1} \rightarrow d\underline{x}_{b/a}^{B}$, $\Delta \underline{\chi}_{b/a}^{B} \rightarrow d\underline{\chi}_{b/a}^{B}$, and $\Delta \underline{\theta}_{IB}^{B} \rightarrow d\underline{\theta}_{IB}^{B}$. Then (12) becomes

$$d\underline{x}^B_{p/a} = d\underline{\chi}^B_{p/a} + d\underline{\theta}^B_{IB} \times \underline{x}^B_{p/a} \qquad d\underline{x}^B_{b/a} = d\underline{\chi}^B_{b/a} + d\underline{\theta}^B_{IB} \times \underline{x}^B_{b/a} \tag{13}$$

The same process yields for the point *b* observed position changes

$$d\underline{x}^B_{p/b} = d\underline{\chi}^B_{p/b} + d\underline{\theta}^B_{IB} \times \underline{x}^B_{p/b} \qquad d\underline{x}^B_{a/b} = d\underline{\chi}^B_{a/b} + d\underline{\theta}^B_{IB} \times \underline{x}^B_{a/b} \tag{14}$$

Similar to (2), we can also define the relative linear velocity between observation points *a* and *b*,

$$\underline{V}^B_{b/a} \equiv \frac{d\underline{\chi}^B_{b/a}}{dt_a} \qquad \underline{V}^B_{a/b} \equiv \frac{d\underline{\chi}^B_{a/b}}{dt_b} \tag{15}$$

where in rotating *B* frame coordinates, $\underline{V}^B_{b/a}$ is the instantaneous velocity of point *b* observed at point *a*, and similarly for $\underline{V}^B_{a/b}$ at point *b*. Dividing the $d\underline{x}^B_{b/a}$ and $d\underline{x}^B_{a/b}$ expressions in (13) and (14) by the corresponding time interval and applying (15) obtains

$$\frac{d\underline{x}^B_{b/a}}{dt_a} = \underline{V}^B_{b/a} + d\underline{\theta}^B_{IB} \times \underline{x}^B_{b/a} \,/\, dt_a \qquad \frac{d\underline{x}^B_{a/b}}{dt_b} = \underline{V}^B_{a/b} + d\underline{\theta}^B_{IB} \times \underline{x}^B_{a/b} \,/\, dt_b \tag{16}$$

Applying the (2) velocity definitions for $\frac{d\underline{x}^B_{b/a}}{dt_a}$ and $\frac{d\underline{x}^B_{a/b}}{dt_b}$, (16) becomes

$$\frac{d\underline{x}^B_{b/a}}{dt_a} = \underline{v}^B_{b/a} = \underline{V}^B_{b/a} + d\underline{\theta}^B_{IB} \times \underline{x}^B_{b/a} \,/\, dt_a \qquad \frac{d\underline{x}^B_{a/b}}{dt_b} = -\,\underline{v}^B_{b/a} = \underline{V}^B_{a/b} + d\underline{\theta}^B_{IB} \times \underline{x}^B_{a/b} \,/\, dt_b \tag{17}$$

where $\underline{v}^B_{b/a}$ is now more specifically defined as the instantaneous velocity of point *b* relative to point *a* as measured in a non-rotating coordinate frame that is instantaneously aligned with the rotating *B* frame.

The first expression in (17) also shows with (2) that

$$\begin{aligned} v^2_{ab} &= \underline{v}^B_{b/a} \cdot \underline{v}^B_{b/a} = \left(\underline{V}^B_{b/a} + d\underline{\theta}^B_{IB} \times \underline{x}^B_{b/a} \,/\, dt_a\right) \cdot \left(\underline{V}^B_{b/a} + d\underline{\theta}^B_{IB} \times \underline{x}^B_{b/a} \,/\, dt_a\right) \\ &= V^2_{b/a} + 2\,\underline{V}^B_{b/a} \cdot \left(d\underline{\theta}^B_{IB} \times \underline{x}^B_{b/a}\right) / \, dt_a + \left(d\underline{\theta}^B_{IB} \times \underline{x}^B_{b/a}\right) \cdot \left(d\underline{\theta}^B_{IB} \times \underline{x}^B_{b/a}\right) / \left(dt_a\right)^2 \end{aligned} \tag{18}$$

where $V_{b/a}$ is the magnitude of $\underline{V}_{b/a}$ in any *B* frame rotating through $d\underline{\theta}^B_{IB}$ relative to non-rotating inertial space. Similarly, the second expression in (17) shows that

$$v_{ab}^2 = \underline{v}_{a/b}^B \cdot \underline{v}_{a/b}^B = \left(\underline{V}_{a/b}^B + d\underline{\theta}_{IB}^B \times \underline{x}_{a/b}^B / dt_b\right) \cdot \left(\underline{V}_{a/b}^B + d\underline{\theta}_{IB}^B \times \underline{x}_{a/b}^B / dt_b\right)$$
$$= V_{a/b}^2 + 2\,\underline{V}_{a/b}^B \cdot \left(d\underline{\theta}_{IB}^B \times \underline{x}_{a/b}^B\right) / dt_b + \left(d\underline{\theta}_{IB}^B \times \underline{x}_{a/b}^B\right) \cdot \left(d\underline{\theta}_{IB}^B \times \underline{x}_{a/b}^B\right) / \left(dt_b\right)^2 \qquad (19)$$

The pertinent results from (2), (13), (14), (15), (17), (18), and (19) summarize as follows

$$\underline{v}_{b/a}^B \equiv \frac{d\underline{x}_{b/a}^B}{dt_a} \qquad \underline{v}_{a/b}^B \equiv \frac{d\underline{x}_{a/b}^B}{dt_b} \qquad \underline{v}_{a/b}^B = -\underline{v}_{b/a}^B$$
$$\underline{V}_{b/a}^B \equiv \frac{d\underline{\chi}_{b/a}^B}{dt_a} \qquad \underline{V}_{a/b}^B \equiv \frac{d\underline{\chi}_{a/b}^B}{dt_b}$$
$$d\underline{x}_{p/a}^B = d\underline{\chi}_{p/a}^B + d\underline{\theta}_{IB}^B \times \underline{x}_{p/a}^B \qquad d\underline{x}_{p/b}^B = d\underline{\chi}_{p/b}^B + d\underline{\theta}_{IB}^B \times \underline{x}_{p/b}^B \qquad (20)$$
$$\underline{v}_{b/a}^B = \underline{V}_{b/a}^B + d\underline{\theta}_{IB}^B \times \underline{x}_{b/a}^B / dt_a \qquad \underline{v}_{a/b}^B = \underline{V}_{a/b}^B + d\underline{\theta}_{IB}^B \times \underline{x}_{a/b}^B / dt_b$$
$$v_{ab}^2 = V_{b/a}^2 + 2\,\underline{V}_{b/a}^B \cdot \left(d\underline{\theta}_{IB}^B \times \underline{x}_{b/a}^B\right) / dt_a + \left(d\underline{\theta}_{IB}^B \times \underline{x}_{b/a}^B\right) \cdot \left(d\underline{\theta}_{IB}^B \times \underline{x}_{b/a}^B\right) / \left(dt_a\right)^2$$
$$= V_{a/b}^2 + 2\,\underline{V}_{a/b}^B \cdot \left(d\underline{\theta}_{IB}^B \times \underline{x}_{a/b}^B\right) / dt_b + \left(d\underline{\theta}_{IB}^B \times \underline{x}_{a/b}^B\right) \cdot \left(d\underline{\theta}_{IB}^B \times \underline{x}_{a/b}^B\right) / \left(dt_b\right)^2$$

Equations (20) can now be substituted into (1) to find differential point-to-point Relativity conversion equations in rotating coordinates.

CONVERTING TO AND FROM ROTATING COORDINATES

Three types of conversion formulas can be defined, 1) Converting rotating coordinate frame effects into non-rotating coordinates, 2) Converting rotating coordinate frame effects viewed by observer *a* with their equivalent rotating coordinate frame effects viewed by observer *b* (and the converse for rotating observer *b* effects viewed by rotating observer *a*), and 3) Converting non-rotating coordinate frame effects into rotating coordinates. The 3) conversion equations are the simplest and will be presented last. The 1) and 2) conversion equations are complex, but analytically straight-forward in their derivation. The method of deriving the 1) and 2) formulas will only be outlined.

To convert rotating coordinate frame effects into non-rotating coordinates, maintain the left side of (1) as shown, and substitute the following expressions from (20) on the right side: $d\underline{x}_{p/a}^B$, $d\underline{x}_{p/b}^B$, $\underline{v}_{b/a}^B$, $\underline{v}_{a/b}^B$, v_{ab}^2. To convert rotating coordinate frame effects observed by one observer into rotating coordinates observed by the other observer, substitute the previous expressions from (20) in both the left and right side of (1).

To convert non-rotating coordinate frame effects into rotating coordinates, maintain the right side of (1) as shown, and substitute the $d\underline{x}_{p/a}^B$ and $d\underline{x}_{p/b}^B$ expressions from (20) on the left.

Having the $d\underline{x}^B_{p/a}$ and $d\underline{x}^B_{p/b}$ conversion formulas in (20) defined in the *B* frame, and the definition for rotating *B* frame coordinates previously defined as being instantaneously aligned with a non-rotating *B* frame, the *B* frame subscript notation in the conversion formulas can be eliminated yielding for the final result:

$$\begin{aligned}
& d\underline{\chi}_{p/a} + d\underline{\theta} \times \underline{x}_{p/a} \\
&= d\underline{x}_{p/b} - \underline{v}_{a/b}\, dt_b + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(d\underline{x}_{p/b} \cdot \underline{v}_{a/b}\, \underline{v}_{a/b} / v_{ab}^2 - \underline{v}_{a/b}\, dt_b \right) \\
& dt_a = \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(dt_b - d\underline{x}_{p/b} \cdot \underline{v}_{a/b} / c^2 \right) \\
& d\underline{\chi}_{p/b} + d\underline{\theta} \times \underline{x}_{p/b} \\
&= d\underline{x}_{p/a} - \underline{v}_{b/a}\, dt_a + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 - \underline{v}_{b/a}\, dt_a \right) \\
& dt_b = \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} \left(dt_a - d\underline{x}_{p/a} \cdot \underline{v}_{b/a} / c^2 \right)
\end{aligned} \tag{21}$$

In (21), $d\underline{\theta}$ vector represents that incremental angular rotation of the rotating coordinate frame relative to non-rotating inertial space.

AN IMPORTANT SPECIAL CASE

As a special case, consider having points *b* and *p* overlap at the start of the *dt* interval (i.e., at t_1). Then $\underline{x}_{p/b} = 0$ and the $d\underline{\chi}_{p/b}$ equation in (21) simplifies to

$$d\underline{\chi}_{p/b} = d\underline{x}_{p/a} - \underline{v}_{b/a}\, dt_a + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1 \right) \left(d\underline{x}_{p/a} \cdot \underline{v}_{b/a}\, \underline{v}_{b/a} / v_{ab}^2 - \underline{v}_{b/a}\, dt_a \right) \tag{22}$$

To further simplify, let us assume that points *a* and *b* are fixed to a rotating rigid body and that the rotating coordinate frame is rotating at the rigid body rotation rate. Then $\underline{V}_{b/a}$ in (20) would be zero and the $\underline{v}_{b/a}$, v_{ab}^2 expressions in (20) become

$$\underline{v}_{b/a} = d\underline{\theta} \times \underline{x}_{b/a} / dt_a \qquad v_{ab}^2 = \left(d\underline{\theta} \times \underline{x}_{b/a} \right) \cdot \left(d\underline{\theta} \times \underline{x}_{b/a} \right) / \left(dt_a \right)^2 \tag{23}$$

Finally, let us look at the case where *p* represents a photon of light. Relativity theory states (as demonstrated for Point-To-Point Relativity kinematics in Appendix A), that in non-rotating inertial space, the speed of light will be constant to any observer, thus

$$d\underline{x}_{p/a} = c\ dt_a\ \underline{u}_{pwg/a} \qquad (24)$$

where $\underline{u}_{pwg/a}$ is a unit vector in the direction of *p* photon travel along a waveguide contained within the rotating structure (as viewed from point *a*). Substituting (23) and (24) into (22) gives

$$\begin{aligned} d\underline{\chi}_{p/b} &= c\ dt_a\ \underline{u}_{pwg/a} - d\underline{\theta} \times \underline{x}_{b/a} \\ &+ \left[\frac{1}{\sqrt{1 - \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \left(d\underline{\theta} \times \underline{x}_{b/a}\right) / \left(c\ dt_a\right)^2}} - 1 \right] \left[\frac{\underline{u}_{pwg/a} \cdot \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \left(d\underline{\theta} \times \underline{x}_{b/a}\right)}{\left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \left(d\underline{\theta} \times \underline{x}_{b/a}\right)} c\ dt_a - d\underline{\theta} \times \underline{x}_{p/a} \right] \end{aligned} \qquad (25)$$

For the common case when the translational motion of point *b* relative to point *a* during dt_a (i.e., $d\underline{\theta} \times \underline{x}_{b/a}$) is much smaller than the distance travelled by a photon over the same time interval ($c\ dt_a$), particular terms in (25) approximate as

$$\begin{aligned} &\sqrt{1 - \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \left(d\underline{\theta} \times \underline{x}_{b/a}\right) / \left(c\ dt_a\right)^2} \\ &\approx 1 - \frac{1}{2} \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \left(d\underline{\theta} \times \underline{x}_{b/a}\right) / \left(c\ dt_a\right)^2 \\ &\frac{1}{\sqrt{1 - \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \left(d\underline{\theta} \times \underline{x}_{b/a}\right) / \left(c\ dt_a\right)^2}} \approx 1 + \frac{1}{2} \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \left(d\underline{\theta} \times \underline{x}_{b/a}\right) / \left(c\ dt_a\right)^2 \\ &\left(\frac{1}{\sqrt{1 - \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \left(d\underline{\theta} \times \underline{x}_{b/a}\right) / \left(c\ dt_a\right)^2}} - 1 \right) \approx \frac{1}{2} \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \left(d\underline{\theta} \times \underline{x}_{b/a}\right) / \left(c\ dt_a\right)^2 \end{aligned} \qquad (26)$$

Substituting the last expression from (26) into (25) yields for $d\underline{\chi}_{p/b}$:

$$\begin{aligned} d\underline{\chi}_{p/b} &= c\ dt_a\ \underline{u}_{pwg/a} - d\underline{\theta} \times \underline{x}_{b/a} \\ &+ \left[\frac{1}{2} \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \left(d\underline{\theta} \times \underline{x}_{b/a}\right) / \left(c\ dt_a\right)^2 \right] \left[\frac{\underline{u}_{pwg/a} \cdot \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \left(d\underline{\theta} \times \underline{x}_{b/a}\right)}{\left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \left(d\underline{\theta} \times \underline{x}_{b/a}\right)} c\ dt_a - d\underline{\theta} \times \underline{x}_{p/a} \right] \end{aligned} \qquad (27)$$

Eq. (27) further simplifies under the previously defined $\left| d\underline{\theta} \times \underline{x}_{b/a} \right| / (c\ dt_a) \ll 1$ common condition. Then terms of second order in $\left| d\underline{\theta} \times \underline{x}_{b/a} \right| / (c\ dt_a)$ are negligible compared with unity and (27) becomes

$$d\underline{\chi}_{p/b} = c\ dt_a\, \underline{u}_{pwg/a} - d\underline{\theta} \times \underline{x}_{b/a} \tag{28}$$

Eq. (28) is the theoretical basis for the operation of optical gyros for which *a* is an arbitrary reference point within the gyro structure, $\underline{x}_{b/a}$ is the distance vector from point *a* to point *b* (another point in the structure), $d\underline{\theta}$ is the incremental rotation of the gyro structure relative to non-rotating inertial space (the signal to be measured by the gyro), point *p* represents a photon of light contained within the gyro structure travelling from point *b*, $d\underline{\chi}_{p/b}$ is the vector distance travelled by the photon from point *b* during the differential time interval dt_a, and $\underline{u}_{pwg/a}$ is a unit vector in the direction of *p* photon travel along a differential segment $d\underline{\chi}_{p/b}$ of a waveguide. For a fiber optic gyro (FOG), the waveguide is a circular coil of optical fiber. For a ring laser gyro (RLG), the waveguide is a closed optical path created by reflecting mirror(s) and an aperture.

THE ANALYTICAL BASIS FOR OPTICAL GYRO DESIGN

The analytical basis for measuring angular rotation in optical gyros is the component of (28) in the direction of photon travel. Relative to a rotating gyro, $d\underline{\chi}_{p/b}$ is constrained to lie along the waveguide, hence relative to point *b* in the waveguide,

$$d\underline{\chi}_{p/b} = dL_p\, \underline{U}_{pwg/b} \tag{29}$$

where dL_p is the differential of distance travelled by photon *p* during time interval dt_a and $\underline{U}_{pwg/b}$ is a unit vector in the instantaneous direction of *p* photon travel at point *b* on the waveguide (as viewed at point *b*). Substituting (29) in (28) and taking the dot product with $\underline{U}_{pwg/b}$ then obtains for dL_p:

$$\begin{aligned} dL_p = d\underline{\chi}_{p/b} \cdot \underline{U}_{pwg/b} &= c_p\ dt_a\, \underline{u}_{pwg/a} \cdot \underline{U}_{pwg/b} - \left(d\underline{\theta} \times \underline{x}_{b/a} \right) \cdot \underline{U}_{pwg/b} \\ &= c_p\ dt_a\, \underline{u}_{pwg/a} \cdot \underline{U}_{pwg/b} - \left(\underline{x}_{b/a} \times \underline{U}_{pwg/b} \right) \cdot d\underline{\theta} \end{aligned} \tag{30}$$

where c_p is the speed of light c in the direction of photon *p* motion within the isotropic medium in which *p* is propagating (that generally differs from the speed of light in open space). Appendix B provides a more rigorous derivation of the dL_p equation giving particular attention to second order effects compared to the first order $\left(\underline{x}_{b/a}\times\underline{U}_{pwg/b}\right)\cdot d\underline{\theta}$ term in (30). Neglecting the second order effects then obtains the simpler but equivalent accuracy dL_p form:

$$dL_p \approx c_p\, dt_a - \left(\underline{x}_{b/a}\times\underline{U}_{pwg/b}\right)\cdot d\underline{\theta} \qquad (31)$$

Comparing (31) with (30) shows that the more rigorous Appendix B derivation effectively sets $\underline{u}_{pwg/a}\cdot\underline{U}_{pwg/b}=1$ in (30) as an approximation having second order error compared to the first order $\left(\underline{x}_{b/a}\times\underline{U}_{pwg/b}\right)\cdot d\underline{\theta}$ term.

The $\underline{x}_{b/a}\times\underline{U}_{pwg/b}$ term in (30) can be further defined by applying (29):

$$\underline{x}_{b/a}\times\underline{U}_{pwg/b} = \underline{x}_{b/a}\times\frac{d\underline{\chi}_{p/b}}{dL_p} = \frac{2\left(\frac{1}{2}\underline{x}_{b/a}\times d\underline{\chi}_{p/b}\right)}{dL_p} \qquad (32)$$

As shown in Fig. 1, the magnitude of $\underline{x}_{b/a}\times d\underline{\chi}_{p/b}/2$ in (32) is the area of the triangle formed between points *a*, *b*, and *p* by $\underline{x}_{b/a}$ and $d\underline{\chi}_{p/b}$:

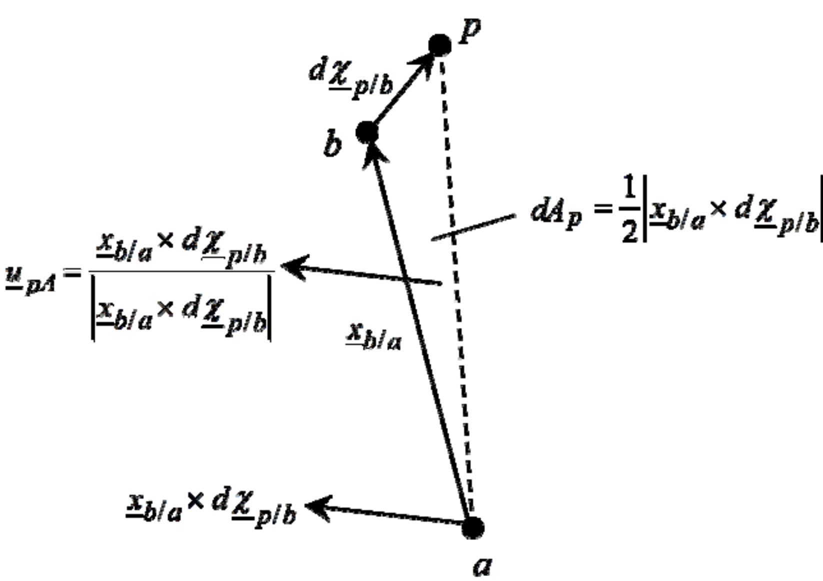

Fig. 1 - Triangular Area Formed By $\underline{x}_{b/a}$ And $d\underline{\chi}_{p/b}$

Designating the magnitude of the $\underline{x}_{b/a} \times d\underline{\chi}_{p/b} / 2$ area vector in Fig. 1 as dA_p obtains

$$\underline{x}_{b/a} \times \underline{U}_{pwg/b} = \frac{2\left(\frac{1}{2}\underline{x}_{b/a} \times d\underline{\chi}_{p/b}\right)}{dL_p} = 2\frac{dA_p}{dL_p}\underline{u}_{pA} \qquad (33)$$

where $\underline{u}_{pA}$ is the unit vector in Fig. 1 parallel to the $\frac{1}{2}\underline{x}_{b/a} \times d\underline{\chi}_{p/b}$ area vector.

Recognizing that the waveguide properties may vary from the isotropic medium nominal, the speed of light c_p in (31) can be expressed as

$$c_p = c_{Nom} + \delta c_p \qquad (34)$$

where c_{Nom} is the nominal speed of light for the isotropic medium and δc_p is the variation from c_{Nom} for photon *p*. With (33) and (34), (31) becomes

$$dL_p = \left(c_{Nom} + \delta c_p\right) dt_a - 2\frac{dA_p}{dL_p}\underline{u}_{pA} \cdot d\underline{\theta} \qquad (35)$$

or with the negative of $\underline{u}_{pA}$ now identified as the instantaneous input axis for *p* photon travel:

$$dL_p = \left(c_{Nom} + \delta c_p\right) dt_a + 2\frac{dA_p}{dL_p} d\theta_{Inpt_p} \qquad (36)$$

where $d\theta_{Inpt_p}$ is the component of $d\underline{\theta}$ along *p* photon input axis $\underline{u}_{Inpt_p}$:

$$d\theta_{Inpt_p} \equiv \underline{u}_{Inpt_p} \cdot d\underline{\theta} = -\underline{u}_{pA} \cdot d\underline{\theta} \qquad (37)$$

Now consider another photon *q* travelling along the same waveguide as photon *p* , but in the opposite direction. Then the equivalent to (28) - (29), (31) - (37), and Fig. 1 over the same dt_a differential time interval used for measuring *p* photon differential travel distance would be

$$d\underline{\chi}_{q/b'} = -c\, dt_a\, \underline{u}_{pwg/a} - d\underline{\theta} \times \underline{x}_{b'/a}$$

$$d\underline{\chi}_{q/b'} = -dL_q\, \underline{U}_{pwg/b'}$$

$$dL_q \approx c_q\, dt_a + \left(\underline{x}_{b'/a} \times \underline{U}_{pwg/b'}\right) \cdot d\underline{\theta}$$

$$\underline{x}_{b'/a} \times \underline{U}_{pwg/b'} = -\underline{x}_{b'/a} \times \frac{d\underline{\chi}_{q/b'}}{dL_q} = -\frac{2\left(\frac{1}{2}\underline{x}_{b'/a} \times d\underline{\chi}_{q/b'}\right)}{dL_q} = 2\frac{dA_q}{dL_q}\underline{u}_{qA} \quad (38)$$

$$c_q = c_{Nom} + \delta c_q$$

$$dL_q = \left(c_{Nom} + \delta c_q\right) dt_a + 2\frac{dA_q}{dL_q}\underline{u}_{qA} \cdot d\underline{\theta}$$

$$dL_q = \left(c_{Nom} + \delta c_p\right) dt_a - 2\frac{dA_q}{dL_q} d\theta_{Inpt_q}$$

$$d\theta_{Inpt_q} \equiv \underline{u}_{Inpt_q} \cdot d\underline{\theta} = -\underline{u}_{qA} \cdot d\underline{\theta}$$

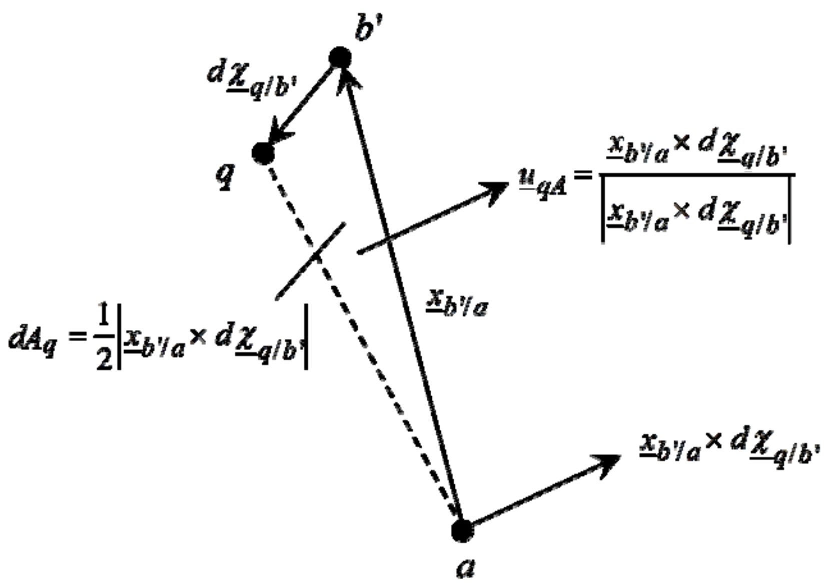

Fig. 2 Triangular Area Formed By $\underline{x}_{b'/a}$ And $d\underline{\chi}_{q/b'}$

In Eqs. (38) and Fig. 2, b' represents another point on the waveguide at the location of photon q, $d\underline{\chi}_{q/b'}$ is the incremental distance vector subtended by photon q along the waveguide relative to the rotating gyro during time increment dt_a, dL_q is the incremental distance travelled by photon q during dt_a, δc_q is the variation in the speed of light from nominal for photon q at point b', dA_q is the area traced out by $d\underline{\chi}_{q/b'}$ relative to point a during dt_a (see Fig. 2 for definition), $\underline{u}_{qA}$ is a unit vector in Fig. 2 perpendicular to dA_q, and θ_{Inpt_q} is the component of $d\underline{\theta}$ along the q photon input axis $\underline{u}_{Inpt_q}$.

The total distances travelled by photons p and q along the waveguide over a general time interval T (measured on a point a clock) is $L_p(T)$ and $L_q(T)$, the integrals of dL_p and dL_q in (36) and (38) over that time interval:

$$
\begin{aligned}
L_p(T) &= \int_{\Delta t_a=0}^{\Delta t_a=T} dL_p = \int_{\Delta t_a=0}^{\Delta t_a=T} \left(c_{Nom} + \delta c_p\right) dt_a + 2\int_{\Delta t_a=0}^{\Delta t_a=T} \frac{dA_p}{dL_p}\, d\theta_{Inpt_p} \\
&= \left(c_{Nom} + \delta c_p\right) T + 2\int_{\Delta t_a=0}^{\Delta t_a=T} \frac{dA_p}{dL_p}\, d\theta_{Inpt_p} \\
L_q(T) &= \int_{\Delta t_a=0}^{\Delta t_a=T} dL_q = \int_{\Delta t_a=0}^{\Delta t_a=T} \left(c_{Nom} + \delta c_q\right) dt_a - 2\int_{\Delta t_a=0}^{\Delta t_a=T} \frac{dA_q}{dL_q}\, d\theta_{Inpt_q} \\
&= \left(c_{Nom} + \delta c_q\right) T - 2\int_{\Delta t_a=0}^{\Delta t_a=T} \frac{dA_q}{dL_q}\, d\theta_{Inpt_q}
\end{aligned}
\tag{39}
$$

Note that each dL_p and dL_q increment accumulated during the (39) integration process has new starting points b (for dL_p) and b' (for dL_q) corresponding to the ending b and b' points of the previous dL_p and dL_q increments. Thus, generalized points b and b' are defined analytically to travel with photons p and q along the waveguide during T. Eq. (39) shows that the effect of input axis rotation ($d\theta_{Inpt_p}$ and $d\theta_{Inpt_q}$) is to generate a difference $\Delta L(T)$ between p and q photon travel distances $L_p(T)$ and $L_q(T)$ of

$$
\begin{aligned}
&\Delta L(T) = L_p(T) - L_q(T) \\
&= \int_{\Delta t_a=0}^{\Delta t_a=T} \left(\delta c_p - \delta c_q\right) dt_a + 2\int_{\Delta t_a=0}^{\Delta t_a=T} \left(\frac{dA_p}{dL_p}\, d\theta_{Inpt_p} + \frac{dA_q}{dL_q}\, d\theta_{Inpt_q}\right)
\end{aligned}
\tag{40}
$$

Measuring $\Delta L(T)$ in (40) provides the mechanism for sensing angular rotation in optical gyros. Note, however, that even though photons p and q travel along the same waveguide, they travel in opposite directions, thus, measuring $\Delta L(T)$ can only be made at a common point in space/time. For optical gyros, this is achieved by forming the waveguide as a closed optical path with photons *p* and *q* beginning and ending their journey around the path at a common point on the waveguide. An optical detector pickoff is located on the gyro at the return point to measure the integrated ΔdL path length difference $\Delta L(T_{Pkf})$ when the photons return (at T_{Pkf}). The path length difference $\Delta L(T_{Pkf})$ manifests itself in a phase shift between the returned *p* and *q* light waves measurable by the pickoff.

A primary goal in optical gyro design is for the $\int_{\Delta t_a=0}^{\Delta t_a=T} \left(\delta c_p - \delta c_q\right) dt_a$ term in (40) around the waveguide to be zero. This is not as simple as it may appear, in part, because although *p* and *q* traverse the same path, they travel in opposite directions, arriving at the same point along the path at different times. For example, if photon *p* is at point *b* at some general time *t* from the starting time, photon *q* will arrive at the same point on the waveguide after an additional $T - t$ seconds have elapsed. The time difference between arrivals would be $\left(T - t\right) - t = T - 2t$ during which the δc_p value may have changed compared with δc_q due to environmental effects. (This is of more concern for fiber optic gyros in which the optical path length of the optical fiber waveguide can be significant.) For an idealized optical gyro, δc_q and δc_p will each be zero for which (40) then simplifies to

$$\Delta L(T) = 2\int_{\Delta t_a=0}^{\Delta t_a=T} \left(\frac{dA_p}{dL_p} \, d\theta_{Inpt_p} + \frac{dA_q}{dL_q} \, d\theta_{Inpt_q} \right) \tag{41}$$

Eq. (41) describes idealized optical gyro operation in general. The form of (41) can be further simplified for certain types of ring laser optical gyros in which the closed waveguide beam path is created by mirror reflected laser beams travelling through an enclosed open space, constrained to remain in a fixed orientation relative to the gyro by an aperture and curved surface mirror(s), and in which the resulting closed beam path lies in a plane. We further stipulate that the beam path be designed for point *a* to represent a centroid that is equally distant along a line perpendicular to any leg (i.e., between any two mirrors) of the beam path. Such a condition would be met for a closed planar light path shaped as a circle, any triangle, or any equal sided tetrahedron having the same angle between any two sides (e.g., a square with 90 degrees between any two sides). For such a closed path configuration, the centroid-to-leg perpendicular distance is exactly $\underline{x}_{b/a} \times \underline{U}_{pwg/b}$ in (33) and $\underline{x}_{b'/a} \times \underline{U}_{pwg/b'}$ in (38). Under these conditions,

$\underline{u}_{pA}, \dfrac{dA_p}{dL_p}, d\theta_{Inpt_p}$ in (32), (33), (37) equals $\underline{u}_{qA}, \dfrac{dA_q}{dL_q}, d\theta_{Inpt_q}$ in (38), and (41) will become

$$L(T) = \int_{\Delta t_a=0}^{\Delta t_a=T} \Delta dL = \int_{\Delta t_a=0}^{\Delta t_a=T} 4\frac{dA}{dL} \, d\theta_{Inpt} = 4\frac{dA}{dL}\int_{\Delta t_a=0}^{\Delta t_a=T} d\theta_{Inpt} \tag{42}$$

where $\dfrac{dA}{dL}$ is the ratio of dA incremental area subtended by a *p* or *q* photon around the waveguide centroid per unit of dL distance travelled during incremental time interval dt_a , and

$d\theta_{Inpt}$ is the corresponding component of gyro incremental angular rotation along the axis perpendicular to the waveguide plane.

Further refinement to (42) is achieved by noting that for T being T_{wg}, the time for photons to traverse the closed waveguide optical path,

$$\int_{\Delta t_a=0}^{\Delta t_a=T_{wg}} \frac{dA}{dL} dL = \int_{\Delta t_a=0}^{\Delta t_a=T_{wg}} dA = A_{wg} \quad \text{and} \quad \int_{\Delta t_a=0}^{\Delta t_a=T_{wg}} \frac{dA}{dL} dL = \frac{dA}{dL} \int_{\Delta t_a=0}^{\Delta t_a=T_{wg}} dL = \frac{dA}{dL} L_{wg} \tag{43}$$

where A_{wg} is the area of the closed optical path and L_{wg} is the closed optical path perimeter. Equating the two expressions in (43) shows that

$$\frac{dA}{dL} = \frac{A_{wg}}{L_{wg}} \tag{44}$$

With (44), (42) assumes the more commonly recognized "area-to-perimeter-ratio" coefficient form

$$\Delta L(T) = 4 \frac{A_{wg}}{L_{wg}} \int_{\Delta t_a=0}^{\Delta t_a=T} d\theta_{Inpt} \tag{45}$$

For the general case where the optical path is not in a single plane, the equivalent to (45) would be (41). At the pickoff, optical gyro measurements corresponding to (41) and (45) would be

For a general waveguide: $$\Delta L(T_{Pkf}) = 2 \int_{\Delta t_a=0}^{\Delta t_a=T_{Pkf}} \left(\frac{dA_p}{dL_p} d\theta_{Inpt_p} + \frac{dA_q}{dL_q} d\theta_{Inpt_q} \right) \tag{46}$$

For a planar waveguide: $$\Delta L(T_{Pkf}) = 4 \frac{A_{wg}}{L_{wg}} \int_{\Delta t_a=0}^{\Delta t_a=T_{Pkf}} d\theta_{Inpt} \tag{47}$$

A fundamental difference in the operation of fiber optic and ring laser gyros is that for a FOG, each photon is generated and input to the waveguide by a super luminescent diode, then traversing the waveguide once to the pickoff. Thus, the pickoff output $\Delta L(T_{Pkf})$ in (46) would be proportional to the integrated angular rate over the time for a single traversal of the photons around the waveguide. In contrast, photons in an RLG are generated by stimulated emission within the waveguide in response to previous photons passing through. This creates the illusion of each photon repeatedly traversing the waveguide, reaching the pickoff at multiple times for each traversal. The result generates a $\Delta L(T_{Pkf})$ output for each traversal representing the running integrated angular rate since gyro turn-on when the photons were initially created.

FUTURE PLANS

A future "Optical Gyros" article planned for the free-access www.strapdownassociates.com website will describe how FOG and RLG optical gyros are configured to implement the optical measurement of $\Delta L(T_{Pkf})$ in (46) in determining for a FOG, the combined $d\theta_{Inpt_p}$, $d\theta_{Inpt_q}$ integrated angular increments along the *p* and *q* photon input axes over the time for a photon to traverse the optical coil, and for a planar RLG, an optical measurement of $\Delta L(T_{Pkf})$ in (47) to determine the integral of $d\theta_{Inpt}$ from gyro turn-on to the current time.

APPENDIX A

Demonstrating With Point-To-Point Relativity Kinematics That Observers Travelling Relative To Each Other Will Measure The Same Speed For An Observed Remote Point Travelling At The Speed Of Light

This appendix demonstrates with Point-To-Point Relativity Eqs. (1) that when a point *p* is travelling at a speed of light velocity as observed at point *a*, the point *p* velocity observed at another point *b* in motion relative to point *a*, will also be at the speed of light. The derivation begins by first defining

$$\underline{v}_{p/b} \equiv \frac{d\underline{x}_{p/b}}{dt_b} \qquad \underline{v}_{p/a} \equiv \frac{d\underline{x}_{p/a}}{dt_a} \tag{A-1}$$

where $\underline{v}_{p/a}$ and $\underline{v}_{p/b}$ are velocities of point *p* determined at observation points *a* and *b*. Dividing the $d\underline{x}_{p/b}$ and dt_b equations in (1) by dt_a obtains with (A-1) and $\underline{v}_{b/a} = v_{ab}\,\underline{u}_v$ from (2):

$$\begin{aligned}
\frac{d\underline{x}_{p/b}}{dt_a} &= \underline{v}_{p/b}\frac{dt_b}{dt_a} = \underline{v}_{p/a} - v_{ab}\,\underline{u}_v + \left(\frac{1}{\sqrt{1 - v_{ab}^2 / c^2}} - 1\right)\left(\underline{v}_{p/a}\cdot\underline{u}_v\,\underline{u}_v - v_{ab}\,\underline{u}_v\right) \\
&= \underline{v}_{p/a} - \underline{v}_{p/a}\cdot\underline{u}_v\,\underline{u}_v + \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}}\left(\underline{v}_{p/a}\cdot\underline{u}_v - v_{ab}\right)\underline{u}_v \\
\frac{dt_b}{dt_a} &= \frac{1}{\sqrt{1 - v_{ab}^2 / c^2}}\left(1 - \underline{v}_{p/a}\cdot\underline{u}_v\,v_{ab} / c^2\right)
\end{aligned} \tag{A-2}$$

The $\underline{v}_{p/a}$ velocity in (A-3) is then defined as having light speed *c* so that

$$\underline{v}_{p/a} = c\,\underline{u}_{p/a} \tag{A-3}$$

where $\underline{u}_{p/a}$ is a unit vector in the direction of $\underline{v}_{p/a}$. Substituting (A-3) in (A-2) obtains

$$\underline{v}_{p/b}\frac{dt_b}{dt_a} = c\left[\underline{u}_{p/a} - \underline{u}_{p/a}\cdot\underline{u}_v\,\underline{u}_v + \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(\underline{u}_{p/a}\cdot\underline{u}_v - v_{ab}/c\right)\underline{u}_v\right]$$

$$\frac{dt_b}{dt_a} = \frac{1}{\sqrt{1-v_{ab}^2/c^2}}\left(1-\underline{u}_{p/a}\cdot\underline{u}_v\,v_{ab}/c\right) \tag{A-4}$$

Recognizing that $\underline{u}_{p/a} - \underline{u}_{p/a}\cdot\underline{u}_v\,\underline{u}_v$ in (A-4) is perpendicular to $\underline{u}_v$ allows application of the Pythagorian Theorem to obtain for the magnitude squared of $\underline{v}_{p/b}\,dt_b/dt_a$:

$$\underline{v}_{p/b}\cdot\underline{v}_{p/b}\left(\frac{dt_b}{dt_a}\right)^2/c^2 = \frac{\left(\underline{u}_{p/a}\cdot\underline{u}_v - v_{ab}/c\right)^2}{\left(1-v_{ab}^2/c^2\right)} + \left(\underline{u}_{p/a} - \underline{u}_{p/a}\cdot\underline{u}_v\,\underline{u}_v\right)\cdot\left(\underline{u}_{p/a} - \underline{u}_{p/a}\cdot\underline{u}_v\,\underline{u}_v\right)$$

$$= \frac{\left(\underline{u}_{p/a}\cdot\underline{u}_v\right)^2 - 2\,\underline{u}_{p/a}\cdot\underline{u}_v\,v_{ab}/c + v_{ab}^2/c^2}{\left(1-v_{ab}^2/c^2\right)} + 1 - \left(\underline{u}_{p/a}\cdot\underline{u}_v\right)^2$$

$$= \frac{\left[\begin{array}{c}\left(\underline{u}_{p/a}\cdot\underline{u}_v\right)^2 - 2\,\underline{u}_{p/a}\cdot\underline{u}_v\,v_{ab}/c + v_{ab}^2/c^2 \\ +1-\left(\underline{u}_{p/a}\cdot\underline{u}_v\right)^2 - v_{ab}^2/c^2 + \left(\underline{u}_{p/a}\cdot\underline{u}_v\right)^2 v_{ab}^2/c^2\end{array}\right]}{\left(1-v_{ab}^2/c^2\right)} \tag{A-5}$$

$$= \frac{-2\,\underline{u}_{p/a}\cdot\underline{u}_v\,v_{ab}/c + 1 + \left(\underline{u}_{p/a}\cdot\underline{u}_v\right)^2 v_{ab}^2/c^2}{\left(1-v_{ab}^2/c^2\right)}$$

$$= \frac{\left(1-\underline{u}_{p/a}\cdot\underline{u}_v\,v_{ab}/c\right)^2}{\left(1-v_{ab}^2/c^2\right)}$$

Identifying the (A-5) result as the square of the dt_b/dt_a term in (A-4) then shows that

$$\underline{v}_{p/b}\cdot\underline{v}_{p/b} = c^2 \tag{A-6}$$

Thus we see from (A-6), that when the magnitude of $\underline{v}_{p/a}$ in (A-3) is the speed of light, the magnitude of $\underline{v}_{p/b}$ will also be the speed of light:

$$\left|\underline{v}_{p/b}\right| = \sqrt{\underline{v}_{p/b}\cdot\underline{v}_{p/b}} = c \tag{A-7}$$

APPENDIX B

A Rigorous Derivation Of The dL_p Equation

This appendix provides a rigorous derivation of Eq. (31) for dL_p. The derivation begins with substitution of $d\underline{\chi}_{p/b}$ from (29) into (28):

$$d\underline{\chi}_{p/b} = dL_p\, \underline{U}_{pwg/b} = c\; dt_a\, \underline{u}_{pwg/a} - d\underline{\theta} \times \underline{x}_{b/a} \tag{B-1}$$

Taking the dot product of (B-1), first with $\underline{U}_{wg/b}$, then with $\underline{u}_{wg/a}$, obtains:

$$\begin{aligned} dL_p &= c\; dt_a\;\; \underline{u}_{pwg/a} \cdot \underline{U}_{pwg/b} - \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \underline{U}_{pwg/b} \\ dL_p\, \underline{U}_{pwg/b} \cdot \underline{u}_{pwg/a} &= c\; dt_a - \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \underline{u}_{wg/a} \end{aligned} \tag{B-2}$$

Dividing (B-2) by $c\; dt_a$:

$$\begin{aligned} dL_p / \left(c\; dt_a\right) &= \underline{u}_{pwg/a} \cdot \underline{U}_{pwg/b} - \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \underline{U}_{pwg/b} / \left(c\; dt_a\right) \\ \underline{U}_{pwg/b} \cdot \underline{u}_{pwg/a}\, dL_p / \left(c\; dt_a\right) &= 1 - \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \underline{u}_{wg/a} / \left(c\; dt_a\right) \end{aligned} \tag{B-3}$$

The first equation in (B-3) becomes for $\underline{U}_{pwg/b} \cdot \underline{u}_{pwg/a}$:

$$\underline{u}_{pwg/a} \cdot \underline{U}_{pwg/b} = dL_p / \left(c\; dt_a\right) + \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \underline{U}_{pwg/b} / \left(c\; dt_a\right) \tag{B-4}$$

Substituting (B-4) in the second (B-3) expression yields

$$\begin{aligned} &\left[dL_p / \left(c\; dt_a\right) + \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \underline{U}_{pwg/b} / \left(c\; dt_a\right)\right] dL_p / \left(c\; dt_a\right) \\ &= \left[dL_p / \left(c\; dt_a\right)\right]^2 + \left[\left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \underline{U}_{pwg/b} / \left(c\; dt_a\right)\right] dL_p / \left(c\; dt_a\right) \\ &= 1 - \left(d\underline{\theta} \times \underline{x}_{b/a}\right) \cdot \underline{u}_{wg/a} / \left(c\; dt_a\right) \end{aligned} \tag{B-5}$$

Eq. (B-5) is a quadratic in $dL_p / \left(c\; dt_a\right)$ whose classic solution is

$$
\begin{aligned}
dL_p/(c\,dt_a) &= -\left[\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}/(c\,dt_a)\right]/2 \\
&+\sqrt{\left[\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}/(c\,dt_a)\right]^2+4\left[1-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{u}_{wg/a}/(c\,dt_a)\right]}/2 \\
&\approx -\left[\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}/(c\,dt_a)\right]/2+\sqrt{\left[1-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{u}_{wg/a}/(c\,dt_a)\right]} \qquad \text{(B-6)} \\
&\approx -\left[\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}/(c\,dt_a)\right]/2+1-\left[\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{u}_{wg/a}/(c\,dt_a)\right]/2 \\
&= 1-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\frac{\left(\underline{U}_{pwg/b}+\underline{u}_{wg/a}\right)}{2}/(c\,dt_a)
\end{aligned}
$$

But from (B-1),

$$
\underline{u}_{pwg/a}=\left(\underline{U}_{pwg/b}\,dL_p+d\underline{\theta}\times\underline{x}_{b/a}\right)/(c\,dt_a) \qquad \text{(B-7)}
$$

Substituting (B-7) into (B-6) gives

$$
\begin{aligned}
dL_p/(c\,dt_a) &= 1-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\frac{\left[\underline{U}_{pwg/b}+\left(\underline{U}_{pwg/b}\,dL_p+d\underline{\theta}\times\underline{x}_{b/a}\right)/(c\,dt_a)\right]}{2}/(c\,dt_a) \qquad \text{(B-8)} \\
&\approx 1-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}/(2\,c\,dt_a)-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}\,dL_p/\left[2\,(c\,dt_a)^2\right]
\end{aligned}
$$

Multiplying (B-8) by $c\,dt_a$ yields

$$
dL_p=c\,dt_a-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}/2-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}\,dL_p/(2\,c\,dt_a) \qquad \text{(B-9)}
$$

or with rearrangement

$$
dL_p\left[1+\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}/(2\,c\,dt_a)\right]=c\,dt_a-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}/2 \qquad \text{(B-10)}
$$

or

$$
\begin{aligned}
dL_p &= \left[c\,dt_a-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}/2\right]\left[1+\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}/(2\,c\,dt_a)\right]^{-1} \\
&\approx \left[c\,dt_a-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}/2\right]\left[1-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}/(2\,c\,dt_a)\right] \qquad \text{(B-11)} \\
&\approx c\,dt_a-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}/2-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}/2 \\
&= c\,dt_a-\left(d\underline{\theta}\times\underline{x}_{b/a}\right).\underline{U}_{pwg/b}
\end{aligned}
$$

and equivalently,

$$dL_p = c\, dt_a - \left(\underline{x}_{b/a} \times \underline{U}_{pwg/b} \right) \cdot d\underline{\theta} \qquad \text{(B-12)}$$

Eq. (B-12) is the simplified, but equivalent version of (30), shown as (31) in the main text.

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