IMPROVED STRAPDOWN INERTIAL SYSTEM CALIBRATION PROCEDURES

PART 3 - NUMERICAL EXAMPLES

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ABSTRACT

This article is Part 3 of a three part series describing an improved Strapdown Rotation Test (SRT) for calibrating the compensation coefficients in a strapdown inertial measurement unit (IMU). The SRT consists of a set of IMU rotations and processing routines that enable precision measurements of IMU gyro/accelerometer misalignment/scale-factor and accelerometer bias errors, all without precision rotation fixturing. The improved SRT is compatible with a broad range of IMU types from aircraft accuracy inertial navigation systems (INSs) to the latest low cost MEMS variety (Micro-machined Electronic Module System). This Part 3 article provides numerical examples showing how collected SRT rotation test data translates sensor errors into data collection measurements, and the impact of neglecting gyro bias in the SRT sensor error determination process. The results numerically confirm that rotation sequences designed in Part 1 measure the particular sensor error for which they were designed.

FOREWORD

This article is the third in a three part series describing improved strapdown rotation test (SRT) procedures for calibrating a strapdown inertial measurement unit (IMU) containing an orthogonal triad of inertial sensors (gyros and accelerometers), digital processor, associated sensor calibration software, and other computational elements. The improved rotation tests consist of a series of rotation sequences, each designed to measure one of the following errors in sensor calibration coefficients: gyro-to-gyro misalignment, accelerometer-to-gyro misalignment, gyro/accelerometer scale-factor, and accelerometer bias. The first and second articles in the three-part series cover the following topics:

Part 1: Procedures, Rotation Fixtures, And Accuracy Analysis - Describes the general theory for the improved rotation tests, rotation test operations, data collection during test, post-test data processing, rotation test fixture requirements, rotation design for sensor error determination, and sensor error determination accuracy analysis.

Part 2: Analytical Derivations - Derives the Part 1 equations for 1) IMU sensor output data processing, 2) Determining sensor calibration errors from the processed IMU data, and 3) SRT inaccuracies caused by rotation fixture error, IMU mounting misalignment on the rotation fixture, approximations in SRT data analysis equations, and residual gyro biases during the SRT.

NOTATION

The following general notation is used throughout this article.

$\underline{V}$ = Vector without specific coordinate frame designation. A vector is a parameter that has length and direction. Vectors used in the paper are classified as "free vectors", hence, have no preferred location in coordinate frames in which they are analytically described.

$\underline{V}^A$ = Column matrix with elements equal to the projection of $\underline{V}$ on coordinate frame A axes. The projection of $\underline{V}$ on each frame A axis equals the dot product of $\underline{V}$ with a unit vector parallel to that coordinate axis.

$\left(\underline{V}^A \times\right)$ = Skew symmetric (or cross-product) form of $\underline{V}^A$ represented by the square matrix $\begin{bmatrix} 0 & -V_{ZA} & V_{YA} \\ V_{ZA} & 0 & -V_{XA} \\ -V_{YA} & V_{XA} & 0 \end{bmatrix}$ in which V_{XA}, V_{YA}, V_{ZA} are the components of $\underline{V}^A$. The matrix product of $\left(\underline{V}^A \times\right)$ with another A frame vector equals the cross-product of $\underline{V}^A$ with the vector in the A frame, i.e.: $\left(\underline{V}^A \times\right) \underline{W}^A = \underline{V}^A \times \underline{W}^A$.

$C_{A_2}^{A_1}$ = Direction cosine matrix that transforms a vector from its coordinate frame A_2 projection form to its coordinate frame A_1 projection form, i.e., $\underline{V}^{A_1} = C_{A_2}^{A_1} \, \underline{V}^{A_2}$. The columns of $C_{A_2}^{A_1}$ are projections on A_1 axes of unit vectors parallel to A_2 axes. Conversely, the rows of $\mathrm{C}_{\mathrm{A}_2}^{\mathrm{A}_1}$ are projections on A_2 axes of unit vectors parallel to A_1 axes. An important property of $C_{A_2}^{A_1}$ is that it's inverse equals it's transpose.

$\underline{\omega}_{I:A}$ = Angular rotation rate of generalized coordinate frame A relative to inertially non-rotating space ($I:A$ subscript).

$\underline{\omega}_{I:E}$ = Angular rotation rate of the earth relative to inertially non-rotating space ($I:E$ subscript).

$\underline{\omega}_{E:A}$ = Angular rotation rate of generalized coordinate frame A relative to the rotating earth ($E : A$ subscript). Note that $\underline{\omega}_{I:A} = \underline{\omega}_{I:E} + \underline{\omega}_{E:A}$ and equivalently, $\underline{\omega}_{E:A} = \underline{\omega}_{I:A} - \underline{\omega}_{I:E}$.

$\dot{(\,)} = \dfrac{d(\,)}{dt}$ = Derivative of parameter ( ) with respect to time t.

$\widehat{(\,)}$ = Computed or measured value of parameter ( ) that, in contrast with the idealized error free value ( ), contain errors.

COORDINATE FRAMES

The primary coordinate frame used in this article is the IMU fixed *B* frame that is rotated relative to the earth (and inertial space) during each SRT rotation sequence. Other coordinate frames related to *B* are fixed (non-rotating) relative to the earth, most aligned with the *B* frame at the start and end of a rotation sequence, one defined to be aligned with north, east, down coordinates at the test site. Specific definitions for the coordinate frame are as follows:

B = IMU sensor frame that is fixed relative to strapdown inertial sensor input axes, but that rotates relative to the earth during each rotation sequence of the SRT. The angular orientation of the *B* frame relative to sensor axes is arbitrary based on user or traditional preferences.

B_{Strt} = Coordinate frame that is fixed (non-rotating) relative to the earth and aligned with the *B* frame at the start of the rotation sequence. Nominally, one of the B_{Strt} frame axes would be aligned with the local vertical if the IMU being tested is perfectly mounted on an idealized rotation fixture.

B_{End} = Coordinate frame that is fixed (non-rotating) relative to the earth and aligned with the *B* frame at the end of the rotation sequence.

$B_{i,\,Strt}$ = Coordinate frame that is fixed (non-rotating) relative to the earth and aligned with the *B* frame at the start of rotation *i* in a rotation sequence.

$B_{i,End}$ = Coordinate frame that is fixed (non-rotating) relative to the earth and aligned with the *B* frame at the end of rotation *i* in a rotation sequence.

MARS = Designation for a “mean-angular-rate-sensor” *B* frame selection, the orthogonal frame that best fits around the actual strapdown gyro input axes.

NED = Earth fixed coordinate frame with axes aligned to local north, east, down directions.

PARAMETER DEFINITIONS

Parameters used throughout the article are defined as follows:

$\hat{\underline{a}}_{SF}^{B}$ = Actual specific force acceleration vector in B frame coordinates (from the IMU accelerometer triad output).

$\hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}}$, $\left(\hat{\underline{a}}_{SF}^{B_{Strt}}\right)_{StrtAvg}$, $\hat{\underline{a}}_{SF_{End}}^{B_{Strt}}$, $\left(\hat{\underline{a}}_{SF}^{B_{Strt}}\right)_{EndAvg}$ = Average values of $\hat{\underline{a}}_{SF}^{B_{Strt}}$ at the start and end of the SRT rotation sequence (when the IMU is stationary).

$A_{SFSign}^{B_{Strt}}$, $A_{SFSign}^{B_{End}}$ = Diagonal matrix with elements equal to unity magnitude with the sign (plus or minus) of the elements of $\underline{a}_{SF}^{B_{Strt}}$, $\underline{a}_{SF}^{B_{End}}$, the true specific force acceleration vector in B_{Strt} and B_{End} coordinates. Equivalently, because the specific force vector is upward when the IMU is stationary, the diagonal matrix elements of $A_{SFSign}^{B_{Strt}}$, $A_{SFSign}^{B_{End}}$ equal the negative of $\underline{u}_{Dwn}^{B_{Strt}}$, $\underline{u}_{Dwn}^{B_{End}}$.

$\hat{a}_{Strt_{Down}}^{B_{Strt}}$, $\hat{a}_{End_{Down}}^{B_{Strt}}$ = Downward components of stationary acceleration measurements at the start and end of the rotation sequence.

$\delta\hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}}$, $\delta\hat{\underline{a}}_{SF_{End}}^{B_{End}}$ = Errors in the (1) measurements of $\hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}}$ and $\hat{\underline{a}}_{SF_{End}}^{B_{Strt}}$.

$\dot{\beta}_i$ = Angular rate of rotation i.

$\Delta\hat{\underline{a}}_{H}^{B_{Strt}}$ = Horizontal component of the difference between stationary acceleration measurements at the end and start of an SRT rotation sequence.

F_{Meas} = Measurement averaging filter output scale factor (integrated effect of $\varsigma\left(t, t_{Meas_{End}}\right)$). Typical averaging filter algorithms are based on a simple linear average or on an average of successive overlapping averages ("average-of-averages") for which [4, Sect. 8.4] shows that for either, F_{Meas} in equals 1 / 2.

g = Plumb-bob gravity magnitude at the test site.

i = Subscript designating the rotation number in a particular SRT rotation sequence.

I = Identity matrix.

$\kappa_{LinScal}$, κ_{Mis}, κ_{Asym} = Gyro triad linear scale factor error, misalignment, and asymmetrical scale-factor error matrices.

$\underline{\kappa}_{Bias}$ = Gyro triad bias error vector.

κ_i = Gyro *i* bias error (component of $\underline{\kappa}_{Bias}$).

κ_{ii} = Gyro *i* linear scale factor error (component of $\kappa_{LinScal}$).

κ_{iii} = Gyro *i* asymmetric scale factor error (component of κ_{Asym}).

κ_{ij} = Gyro *i* misalignment error coupling angular rate from axis *j* into the gyro *i* input axis (component of κ_{Mis}).

l = Latitude of the test site.

$\underline{\lambda}_{Bias}$ = Accelerometer triad bias error vector.

$\lambda_{LinScal}$, λ_{Mis}, λ_{Asym} = Accelerometer triad linear scale factor, misalignment, and asymmetrical scale-factor error matrices.

λ_i = Accelerometer *i* bias error (component of $\underline{\lambda}_{Bias}$).

λ_{ii} = Accelerometer *i* linear scale factor error (component of $\lambda_{LinScal}$).

λ_{iii} = Accelerometer *i* asymmetric scale factor error (component of λ_{Asym}).

λ_{ij} = Accelerometer *i* misalignment error coupling acceleration from axis *j* into the accelerometer *i* input axis (component of λ_{Mis}).

μ_{ij} = Misalignment of accelerometer *i* relative to *MARS B* frame axis *j*.

n = Subscript designating rotation number *i* for the last rotation in a particular SRT sequence.

ω_e = Magnitude of earth's rotation rate relative to non-rotating inertial space.

$\underline{\phi}_{End}^{BStrt}$ = Rotation angle error vector imbedded within the (2) computed $\hat{C}_B^{BStrt}$ at the end of the rotation sequence.

θ_i = Signed magnitude of total angular traversal around rotation axis i.

$t_{Meas_{Strt}}$, $t_{Meas_{End}}$ = Time at the start and end of the $\underline{\hat{a}}^{B_{Strt}}$ measurement time.

T_{Meas} = Time interval for making each of the $\underline{\hat{a}}^{B_{Strt}}$ SRT measurements at the start and end of the rotation sequence.

$t_{SeqStrt}$ = Time at the start of the first stationary acceleration measurement averaging process for the rotation sequence.

$\underline{\hat{u}}_{Dwn}^{B_{Strt}}$ = Estimated unit vector downward (along plumb-bob gravity) in B_{Strt} frame coordinates; e.g., $\underline{\hat{u}}_{Dwn}^{B_{Strt}} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T, \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$, or $\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$ for IMU axis x, y, or z downward at the start of a rotation sequence.

$\underline{u}_{Dwn}^{B_{Strt}}, \underline{u}_{Dwn}^{B_{End}}$ = Unit vectors downward in the B_{Strt} and B_{End} frames.

$\underline{u}_{Dwn}^{NED}$ = Unit vector downward (along plumb-bob gravity) in *NED* frame coordinates, e.g., for local down along the NED third (e.g., z) axis, $\underline{u}_{Dwn}^{NED} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$.

$\underline{u}_i^{B_{i,Strt}}$ = Unit vector along the rotation axis of rotation i in the rotation sequence, also defined for the SRT to be a along a particular IMU B frame axis; e.g., $\underline{u}_i^{B_{i,Strt}} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T, \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T$, or $\begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$ for rotation i around B frame axis x, y, or z.

υ_{ij} = Orthogonality error between gyro axes i and j.

$\Delta\underline{\phi}_{GyroBiasRot}^{B_{Strt}}$ = Error in $\underline{\phi}_{End}^{B_{Strt}}$ caused by neglecting the effect of gyro bias during an SRT rotation sequence.

$e\left(\Delta\underline{\hat{a}}_H^{B_{Strt}}\right)_{GyroBias}$ = Error in the (5) error model for $\Delta\underline{\hat{a}}_H^{B_{Strt}}$ caused by neglecting the effect of gyro bias in $\underline{\phi}_{End}^{B_{Strt}}$.

$\varsigma\left(t, t_{Meas_{End}}\right)$ = Measurement averaging algorithm weighting function: The algorithm response at $t_{Meas_{End}}$ to a unit impulse input to the averaging algorithm at time t during the measurement period.

INTRODUCTION

The Strapdown Rotation Test (SRT) is a test procedure designed for rapid measurement/calibration of sensor error parameters in a strapdown inertial measurement unit (IMU): gyro/accelerometer scale factor errors, accelerometer bias errors, and sensor-to-sensor misalignment errors. The SRT is performed by executing a series of IMU rotation sequences with the IMU mounted on a two-axis rotation fixture. The principal advantage of the SRT is the ability to precisely determine sensor errors (e.g., misalignments to micro-radian accuracy) using moderate accuracy (one milli-radian) rotation test fixturing.

The original SRT concept was first disclosed in 1977 [1], based on defining the test measurement as the rate of change of horizontal velocity generated within the IMU from a standard inertial navigation solution. Measurements were taken with the IMU stationary between SRT rotation sequence executions. Under stationary conditions, an IMU with ideal (perfectly calibrated) inertial sensors has zero horizontal velocity. Non-zero stationary horizontal velocity rates provide measures of sensor errors excited by the IMU rotation sequences and reorientation from sequence start. Analytical routines within the SRT translated the stationary horizontal velocity-rate measurements into the sensor errors that created them.

The [1] method evolved to the form disclosed in 2002 [2, Sect. 18.4] whereby the velocity change measurement was replaced by the difference between horizontal acceleration measurements taken at the start and end of each SRT rotation sequence. The horizontal acceleration measurements were averaged outputs from a strapdown “analytical platform” generated using the IMU strapdown gyro/accelerometer sensor signals. (An analytic platform is a fundamental computational element in a strapdown IMU that transforms strapdown accelerometer signals through a direction cosine matrix (DCM) into a non-rotating reference coordinate frame, analogous to a mechanically gimbaled gyro-stabilized platform on which the accelerometers and gyros would be mounted). For the SRT, the analytic platform resides in the IMU under test or as software in the SRT test computer. The latter concept is depicted in Fig. 1, the DCM being calculated in the “Attitude Computation” block.

The latest form of the SRT disclosed in Part 1 of the three part series eliminates the [1] and [2] requirement for IMU inertial-sensor-based initial “self-alignment” of the Figure 1 DCM at SRT start, thereby expanding SRT applicability to IMUs having sensors without the required accuracy for accurate DCM inertial alignment - e.g., MEMS (micro-machined electronic system) IMUs. Additionally, each rotation sequence in the improved SRT is designed to measure a particular sensor error parameter, thus simplifying data processing, and reducing the time for potential sensor error shifts between measurements (compared to the original [1] and [2] concepts where the time for potential sensor error shifts spanned initial DCM alignment and several rotation sequences).

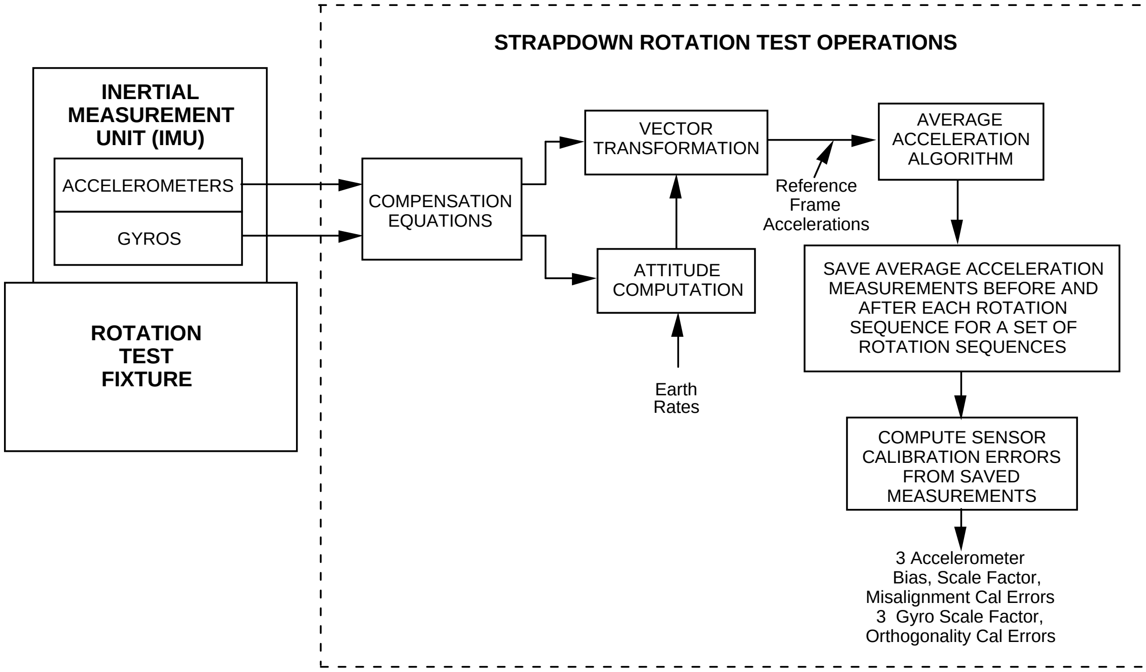

Fig. 1 - Strapdown Rotation Test (SRT) Setup

The rotation sequences for the improved SRT are designed in Part 1 [3, Section 6.0] for execution using a two-axis rotation fixture with outer axis horizontal. Table 1 defines the sequences for an IMU mounted on the rotation fixture with z axis (of a right-handed mutually orthogonal x, y, z set) aligned with the inner rotation axis and downward when the outer axis rotation angle is zero. The IMU x, y axis mounting is defined as having the y axis aligned with the outer rotation fixture axis when the inner axis rotation angle is zero.

Table 1 is a specialized version of five generic rotation sequence groups, each designed in Part 1 [3, Sect. 6.0] to excite a particular sensor error type onto the SRT measurement, 1) Gyro scale factor error [3, Sect. 6.2.1], 2) Gyro-to-gyro orthogonality error [3, Sect. 6.2.2.1], 3) Gyro-to-gyro orthogonality error when circumventing mechanical limitations of a two-axis rotation fixture [3, Sect. 6.2.2.2], 4) Accelerometer misalignment error relative to gyro triad input axes [3, Sect. 6.3.1.1], and 5) Accelerometer bias error [3, Sect.6.3.1.2]. For the IMU x, y, z axis setup and rotation sequence definitions, Table 1 rotations sequences contained within the previously defined five groups are as follows: 1) Sequences 1 – 3 for x, y, z gyro scale factor error, 2) Sequences 4 -5 for y to z and z to x gyro-to-gyro orthogonality error, 3) Sequence 6 for x to y gyro orthogonality error, 4) Sequences 7 – 12 for accelerometer-to-gyro misalignment error (x accelerometer to y gyro, x accelerometer to z gyro, z to x, z to y, y to z, and y to x), and 5) Sequences 13 – 14 for x, y accelerometer bias - z accelerometer bias is obtained from the group four sequence 7 measurement component not containing accelerometer-to-gyro misalignment.

This Part 3 article first summarizes the improved SRT methodology, and then provides five detailed numerical examples, each deriving the measurement obtained for a particular Table 1 rotation sequence, each being a sample of the previously defined five sensor error determination sequence groups: 1) Sequence 3 for z gyro scale factor error, 2) Sequence 5 for z to x gyro orthogonality error, 3) Sequence 6 for x to y gyro orthogonality error, 4) Sequence 7 for misalignment error between accelerometer x and gyro y (on one of the sequence 7 measurement horizontal components, z accelerometer bias on the other), and 5) Sequence 13 for y accelerometer bias. Numerical results obtained are identical to those derived in the generalized analytical Part 1 [3, Section 6.0] design of the rotation sequences. Included for each rotation sequence example is a numerical evaluation of the impact of neglecting gyro bias in the improved SRT processing equations.

Sequence Number	Initial IMU Axis Directions Down	Initial IMU Axis Directions Along Outer Rotation Axis	Initial Rotation Fixture Angles Inner	Initial Rotation Fixture Angles Outer	Sequential IMU Axis Rotations
1	Z	Y	0	0	+360 Y
2	Z	Y	0	0	+360 X
3	X	Y	0	-90	+360 Z
1a	Z	Y	0	0	-360 Y
2a	Z	Y	0	0	-360 X
3a	X	Y	0	-90	-360 Z
4	Z	Y	0	0	+180 Y, +180 Z , +180 Y, +180 Z
5	Z	X	+90	0	+180 X, +180 Z , +180 X, +180 Z
6	X	Y	0	-90	+180 Y, +90 Z, +180 X, +90 Z, +180 Y, +90 Z, +180 X, +90 Z
7	Y	X	+90	90	+180 X
8	Z	X	+90	0	+180 X
9	X	Y	0	-90	+180 Z
10	Y	X	+90	90	+180 Z
11	Z	Y	0	0	+180 Y
12	X	Y	0	-90	+180 Y
13	Z	Y	0	0	+180 Z, +180 Y
14	Z	X	+90	0	+180 Z, +180 X

*Note - Rotation sequences 1a - 3a are not needed when gyros have no scale factor asymmetry.

Table 1 - Improved Strapdown Rotation Test Sequences

STRAPDOWN ROTATION TEST DATA COLLECTION

For each SRT rotation sequence, the following operations from Part 1 [3, Eqs. (3)] would be performed prior to and after completion of the sequence rotations to obtain the "Reference Frame Accelerations" in Fig. 1:

$$
\begin{gathered}
\hat{\underline{a}}_{SF}^{B_{Strt}} = \hat{C}_{B}^{B_{Strt}}\, \hat{\underline{a}}_{SF}^{B} \\
\hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}} \equiv \left(\hat{\underline{a}}_{SF}^{B_{Strt}}\right)_{StrtAvg} \qquad \hat{\underline{a}}_{SF_{End}}^{B_{Strt}} \equiv \left(\hat{\underline{a}}_{SF}^{B_{Strt}}\right)_{EndAvg} \\
\Delta\hat{\underline{a}}_{H}^{B_{Strt}} = \left(\hat{\underline{a}}_{SF_{End}}^{B_{Strt}} - \hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}}\right)_{H} \\
\hat{\underline{u}}_{Dwn}^{B_{Strt}} = \hat{C}_{NED}^{B_{Strt}}\, \underline{u}_{Dwn}^{NED} \qquad \underline{u}_{Dwn}^{NED} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{T} \\
\hat{a}_{Strt_{Down}}^{B_{Strt}} = \hat{\underline{u}}_{Dwn}^{B_{Strt}} \cdot \hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}} + g \qquad \hat{a}_{End_{Down}}^{B_{Strt}} = \hat{\underline{u}}_{Dwn}^{B_{Strt}} \cdot \left(\hat{\underline{a}}_{SF_{End}}^{B_{Strt}}\right) + g
\end{gathered}
\tag{1}
$$

The components of $\hat{\underline{a}}_{SF}^{B_{Strt}}$ in (1) represent the "Reference Frame Accelerations" in Fig. 1, the reference frame being the *B* frame at the start of the sequence (i.e., B_{Strt}). The $\left(\hat{\underline{a}}_{SF}^{B_{Strt}}\right)_{StrtAvg}$, $\left(\hat{\underline{a}}_{SF}^{B_{Strt}}\right)_{EndAvg}$ components in (1) represent outputs from the "Average Acceleration Algorithm" block in Fig. 1 calculated from the average value of $\hat{\underline{a}}_{SF}^{B_{Strt}}$ over a designated time period at the start and end of the rotation sequence. The average acceleration measurements typically last for 10 seconds each using a simple averaging or average-of-averages type algorithm. The $\hat{C}_{NED}^{B_{Strt}}$ matrix in (1) is the orientation of the IMU *B* frame relative to local NED (north, east, down) coordinates at the start of the rotation sequence, approximately known from the rotation fixture north orientation in the test facility and the IMU mounting orientation on the text fixture. The $\hat{C}_{B}^{B_{Strt}}$ matrix in (1) is the output of the Fig. 1 "Attitude Computation" block, calculated from Part 1, [3, Eqs. (4)], as an integration process from the start of each rotation sequence:

$$
\begin{gathered}
\dot{\hat{C}}_{B}^{B_{Strt}} = \hat{C}_{B}^{B_{Strt}}\left(\hat{\underline{\omega}}_{I:B}^{B}\times\right) - \left(\hat{\underline{\omega}}_{I:E}^{B_{Strt}}\times\right)\hat{C}_{B}^{B_{Strt}} \\
\hat{\underline{\omega}}_{I:E}^{B_{Strt}} = \hat{C}_{NED}^{B_{Strt}}\, \underline{\omega}_{I:E}^{NED} \qquad \underline{\omega}_{I:E}^{NED} = \begin{bmatrix} \omega_e \cos l & 0 & -\omega_e \sin l \end{bmatrix}^{T} \\
\hat{C}_{B}^{B_{Strt}} = I + \int_{t_{SeqStrt}}^{t} \dot{\hat{C}}_{B}^{B_{Strt}}\, dt
\end{gathered}
\tag{2}
$$

Note in (2) that the $\hat{C}_B^{B_{Strt}}$ matrix is initialized at identity, thus designating the B fame at the start of the sequence as the reference frame in Fig. 1 for making rotation sequence "Reference Frame Acceleration" measurements.

DETERMINING IMU SENSOR ERRORS

Approximate error models are derived in Part 2 [4, Sect. 7.2] and summarized in Part 1 [3, Eqs. (5) – (7)] defining the $\Delta\hat{\underline{a}}_H^{B_{Strt}}$, $\hat{a}_{Down}^{B_{Strt}}$, and $\hat{a}_{Down}^{B_{End}}$ measurements in (1) as a function of individual gyro and accelerometer error parameters for <u>each</u> rotation sequence in the SRT:

$$
\begin{gathered}
\underline{\phi}_{End}^{B_{Strt}} \approx \sum_i C_{B_{i,Strt}}^{B_{Strt}} \left\{ \begin{array}{c} \left[\kappa_{LinScal} + \kappa_{Asym}\, \mathrm{Sign}\left(\dot{\beta}_i\right)\right] \underline{u}_i^{B_{i,Strt}}\, \theta_i \\ + \left[I \sin\theta_i + \left(1 - \cos\theta_i\right)\left(\underline{u}_i^{B_{i,Strt}} \times\right)\right] \left(\kappa_{Mis}\, \underline{u}_i^{B_{i,Strt}}\right) \end{array} \right\} \\
C_{B_{1,Strt}}^{B_{Strt}} = I \qquad \text{Do } i = 1 \text{ To } n:\ C_{B_{i+1,Strt}}^{B_{Strt}} = C_{B_{i,Strt}}^{B_{Strt}}\, C_{B_{i+1,Strt}}^{B_{i,Strt}} \\
C_{B_{i+1,Strt}}^{B_{i,Strt}} = I + \sin\theta_i \left(\underline{u}_i^{B_{i,Strt}} \times\right) + \left(1 - \cos\theta_i\right) \left(\underline{u}_i^{B_{i,Strt}} \times\right)^2 \\
C_{B_{End}}^{B_{Strt}} = C_{B_{n+1,Strt}}^{B_{Strt}}
\end{gathered} \tag{3}
$$

$$
\begin{aligned}
\delta\hat{\underline{a}}_{SF\,Strt}^{B_{Strt}} &\approx - g\left(\lambda_{LinScal} + \lambda_{Mis} + \lambda_{Asym}\, A_{SFSign}^{B_{Strt}}\right) \underline{u}_{Dwn}^{B_{Strt}} + \underline{\lambda}_{Bias} \\
\delta\hat{\underline{a}}_{SF\,End}^{B_{End}} &\approx - g\left(\lambda_{LinScal} + \lambda_{Mis} + \lambda_{Asym}\, A_{SFSign}^{B_{End}}\right) \underline{u}_{Dwn}^{B_{End}} + \underline{\lambda}_{Bias}
\end{aligned} \tag{4}
$$

$$
\begin{gathered}
\Delta\hat{\underline{a}}_H^{B_{Strt}} \approx g\, \underline{u}_{Dwn}^{B_{Strt}} \times \underline{\phi}_{End}^{B_{Strt}} + \left(C_{B_{End}}^{B_{Strt}}\, \delta\hat{\underline{a}}_{SF\,End}^{B_{End}} - \delta\hat{\underline{a}}_{SF\,Strt}^{B_{Strt}}\right)_H \\
\hat{a}_{Strt\,Down}^{B_{Strt}} \approx \underline{u}_{Dwn}^{B_{Strt}} \cdot \delta\hat{\underline{a}}_{SF\,Strt}^{B_{Strt}} \qquad \hat{a}_{End\,Down}^{B_{End}} \approx \underline{u}_{Dwn}^{B_{End}} \cdot \delta\hat{\underline{a}}_{SF\,End}^{B_{End}} \\
\underline{u}_{Dwn}^{B_{End}} = \left(C_{B_{End}}^{B_{Strt}}\right)^T \underline{u}_{Dwn}^{B_{Strt}}
\end{gathered} \tag{5}
$$

Equating the (1) measurements to the equivalent (5) error model for each rotation sequence provides a simultaneous set of linear equations that can be inverted to determine the sensor error parameters.

Elements within the (3) - (4) error matrices and vectors are defined as

$$\kappa_{LinScal} = \begin{bmatrix} \kappa_{xx} & 0 & 0 \\ 0 & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{bmatrix} \quad \kappa_{Mis} = \begin{bmatrix} 0 & \kappa_{xy} & \kappa_{xz} \\ \kappa_{yx} & 0 & \kappa_{yz} \\ \kappa_{zx} & \kappa_{zy} & 0 \end{bmatrix} \quad \kappa_{Asym} = \begin{bmatrix} \kappa_{xxx} & 0 & 0 \\ 0 & \kappa_{yyy} & 0 \\ 0 & 0 & \kappa_{zzz} \end{bmatrix} \tag{6}$$

$$\begin{gathered} \lambda_{LinScal} = \begin{bmatrix} \lambda_{xx} & 0 & 0 \\ 0 & \lambda_{yy} & 0 \\ 0 & 0 & \lambda_{zz} \end{bmatrix} \quad \lambda_{Mis} = \begin{bmatrix} 0 & \lambda_{xy} & \lambda_{xz} \\ \lambda_{yx} & 0 & \lambda_{yz} \\ \lambda_{zx} & \lambda_{zy} & 0 \end{bmatrix} \\ \lambda_{Asym} = \begin{bmatrix} \lambda_{xxx} & 0 & 0 \\ 0 & \lambda_{yyy} & 0 \\ 0 & 0 & \lambda_{zzz} \end{bmatrix} \quad \underline{\lambda}_{Bias} = \begin{bmatrix} \lambda_x \\ \lambda_y \\ \lambda_z \end{bmatrix} \end{gathered} \tag{7}$$

The gyro misalignments in (6) are relative to an arbitrary selected coordinate frame *B* representing IMU inertial sensor axes. To minimize second order error effects, it is expeditious to select the *B* frame to correspond with *MARS* (mean angular rate sensor) axes, the orthogonal frame that best fits around the actual gyro input axes. Fig. 3 illustrates the concept.

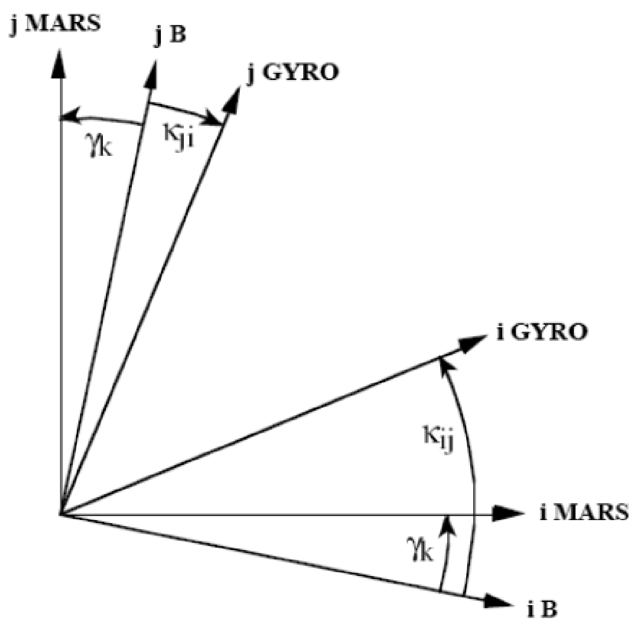

Fig. 3 - *MARS* Coordinates

In Fig. 3, γ_k is the angle between MARS and general *B* frame axes *i* and *j*. From Fig. 3, defining the *B* frame to be a *MARS* type is equivalent to setting $\gamma_k = 0$ for which

$$\kappa_{ij} = \kappa_{ji} \tag{8}$$

When adopting the *MARS* frame for *B*, it is also expedient to redefine κ_{ij} in terms of the angular orthogonality error between *i* and *j*, i.e., the angle between *i* and *j* gyro axes compared with the nominal orthogonal *MARS* axes equivalent of π/2. From Fig. 3, the conversion formula is

$$\upsilon_{ij} = \kappa_{ij} + \kappa_{ji} \tag{9}$$

or with (8),

$$\kappa_{ij} = \kappa_{ji} = \frac{1}{2}\,\upsilon_{ij} \tag{10}$$

For a *MARS* defined *B* frame, the 6 accelerometer λ_{ij} misalignments in (7) will then automatically become *MARS* reference specialized. To identify *MARS* specialization and compatibility with *MARS* referenced gyro misalignments in (10), we will adopt the accelerometer misalignment definition formula

$$\lambda_{ij} = \mu_{ij} \tag{11}$$

IMPACT OF NEGLECTING GYRO BIAS ON SRT ACCURACY

The SRT error model in (3) is based on the assumption that the IMU gyro biases have been calibrated to a reasonable accuracy prior to SRT rotation sequence execution. The SRT measurement error caused by neglecting constant gyro bias is derived as $\Delta\hat{\underline{a}}^{B_{Strt}}_{GyroBias}$ in Part 2 [4, Sect. 8.0] and analyzed in Part 1 [3, Sect 5.2.6]. Approximating the nominal *B* frame at the start of each rotation as general frame *B*, Part 1 [3, Eqs. (32) – (34)] for the gyro bias effect is

$$\Delta\underline{\phi}^{B_{Strt}}_{GyroBiasRot} = \sum_i C^{B_{Strt}}_{B_{i,Strt}} \left[I + \frac{(1-\cos\theta_i)}{\theta_i}\left(\underline{u}_i^{B_{i,Strt}}\times\right) + \left(1 - \frac{\sin\theta_i}{\theta_i}\right)\left(\underline{u}_i^{B_{i,Strt}}\times\right)^2 \right] \frac{\theta_i}{\dot{\beta}_i}\,\underline{\kappa}_{Bias} \tag{12}$$

$$e\left(\Delta\hat{\underline{a}}^{B_{Strt}}\right)_{GyroBias} = g\;\underline{u}^{B_{Strt}}_{Dwn} \times \left\{ \Delta\underline{\phi}^{B_{Strt}}_{GyroBias_{Rot}} + T_{Meas}\left[I + F_{Meas}\left(C^{B_{Strt}}_{B_{End}} - I \right)\right]\underline{\kappa}_{Bias} \right\} \tag{13}$$

$$F_{Meas} \equiv \frac{1}{T_{Meas}} \int_{t_{Meas_{Strt}}}^{t_{Meas_{End}}} \varsigma\left(t, t_{Meas_{End}}\right)\left(t - t_{MeasStrt}\right) dt \tag{14}$$

ROTATION SEQUENCE MEASUREMENT EXAMPLES

This last section in the article, evaluates the (3) – (5) error model equations for particular Table 1 rotation sequences 3, 5, 6, 7 and 13. The evaluation uses numerical values for the matrix and vector parameters in (3) – (5) to generate particular solutions for comparison with the equivalent generated in Part 1 [3, Section 6.0] using a purely analytical development approach. Identical results between the two approaches confirms the accuracy of the general derivation solutions presented in Part 1. Additionally, for each of Table 1 rotation sequence 3, 5, 6, 7, 13, this section also evaluates the (12) – (13) gyro bias effect on SRT accuracy for application in the Part 1 [3, Section 5.2.6] SRT error analysis.

ROTATION SEQUENCE 3

Rotation sequence 3 is representative of sequences 1 – 3 and 1a – 3a used to determine x, y, z gyro linear and asymmetric scale factor errors. From Table 1, sequence 3 is a single 360 degree rotation which returns the IMU to its starting orientation. Then

$$C_{B_{End}}^{B_{Strt}} = I \qquad \delta\underline{\hat{a}}_{SF_{End}}^{B_{End}} = \delta\underline{\hat{a}}_{SF_{Strt}}^{B_{Strt}} \tag{15}$$

Table 1, sequence 3 also shows that IMU axis x is initially down and the IMU z axis is the axis of rotation:

$$\underline{u}_{Dwn}^{B_{Strt}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{u}_{1}^{B_{1,Strt}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{16}$$

<u>Sequence 3 Measurement</u>

With (15), (3) and (5) for the +360 deg rotation simplify to

$$\underline{\phi}_{End}^{B_{Strt}} = 2\pi\left(\kappa_{LinScal} + \kappa_{Asym}\right)\underline{u}_{1}^{B_{1,Strt}} \qquad \Delta\underline{\hat{a}}_{H}^{B_{Strt}} = g\ \underline{u}_{Dwn}^{B_{Strt}} \times \underline{\phi}_{End}^{B_{Strt}} \tag{17}$$

Substituting (16) in (17) with $\kappa_{LinScal}$ and κ_{Asym} from (6) finds

$$\begin{gathered}
\underline{\phi}_{End}^{B_{Strt}} = 2\pi\begin{bmatrix} \kappa_{xx}+\kappa_{xxx} & 0 & 0 \\ 0 & \kappa_{yy}+\kappa_{yyy} & 0 \\ 0 & 0 & \kappa_{zz}+\kappa_{zzz} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2\pi\left(\kappa_{zz}+\kappa_{zzz}\right) \end{bmatrix} \\
\Delta\underline{\hat{a}}_{H}^{B_{Strt}} = g\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 2\pi\left(\kappa_{zz}+\kappa_{zzz}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ -2\pi g\left(\kappa_{zz}+\kappa_{zzz}\right) \\ 0 \end{bmatrix}
\end{gathered} \tag{18}$$

The y component in (18) is identical to that for sequence 3 in Part 1 [3, Eqs. (16)].

Sequence 3 Unmodelled Gyro Bias Error Effect

For the Table 1 sequence 3 single axis +360 degree rotation, (12) and (13) with (15) for $C_{B_{End}}^{B_{Strt}}$ simplify to

$$\Delta\underline{\phi}_{GyroBiasRot}^{B_{Strt}} = \left[I + \left(\underline{u}_1^{B_1,Strt} \times \right)^2 \right] \frac{2\pi}{\dot{\beta}_1} \underline{\kappa}_{Bias}$$
$$e\left(\Delta\hat{\underline{a}}^{B_{Strt}}\right)_{GyroBias} = g\, \underline{u}_{Dwn}^{B_{Strt}} \times \left(\Delta\underline{\phi}_{GyroBias_{Rot}}^{B_{Strt}} + T_{Meas}\, \underline{\kappa}_{Bias} \right) \tag{19}$$

With (16) and (6) for $\underline{\kappa}_{Bias}$, (19) becomes

$$\Delta\underline{\phi}_{GyroBiasRot}^{B_{Strt}} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \frac{2\pi}{\dot{\beta}_1} \begin{bmatrix} \kappa_X \\ \kappa_Y \\ \kappa_Z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2\pi\, \kappa_Z / \dot{\beta}_1 \end{bmatrix}$$

$$e\left(\Delta\hat{\underline{a}}^{B_{Strt}}\right)_{GyroBias} = g \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \left\{ \begin{bmatrix} 0 \\ 0 \\ 2\pi\, \kappa_Z / \dot{\beta}_1 \end{bmatrix} + T_{Meas} \begin{bmatrix} \kappa_X \\ \kappa_Y \\ \kappa_Z \end{bmatrix} \right\} \tag{20}$$
$$= \begin{bmatrix} 0 \\ -2\pi\, g \left[1/\dot{\beta}_1 + T_{Meas}/(2\pi) \right] \kappa_Z \\ g\, T_{Meas}\, \kappa_Y \end{bmatrix}$$

ROTATION SEQUENCE 5

Table 1 rotation sequence 5 is representative of sequences 4 and 5 used to determine the orthogonality error between gyro z and gyros x, y. From Table 1, sequence 5 consists of four +180 degree rotations which return the IMU to its starting orientation. Then

$$C_{B_{End}}^{B_{Strt}} = I \qquad \delta\hat{\underline{a}}_{SF_{End}}^{B_{End}} = \delta\hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}} \tag{21}$$

Table 1 sequence 5 also shows that IMU axis z is initially down and the sequence of four +180 deg rotations is about IMU axes x, z, x, z in that order:

$$\underline{u}_{Dwn}^{B_{Strt}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \underline{u}_1^{B_1,Strt} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \underline{u}_2^{B_2,Strt} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \underline{u}_3^{B_3,Strt} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \underline{u}_4^{B_4,Strt} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{22}$$

From (22) and (3) we can then write:

$$C_{B1,Strt}^{BStrt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{+180 about x:} \quad C_{B2,Strt}^{BStrt} = C_{B1,Strt}^{BStrt}\, C_{B2,Strt}^{B1,Strt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{+180 about z:} \quad C_{B3,Strt}^{BStrt} = C_{B2,Strt}^{BStrt}\, C_{B3,Strt}^{B2,Strt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (23)$$

$$\text{+180 about x:} \quad C_{B4,Strt}^{BStrt} = C_{B3,Strt}^{BStrt}\, C_{B4,Strt}^{B3,Strt} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{+180 about z:} \quad C_{BEnd}^{BStrt} = C_{B4,Strt}^{BStrt}\, C_{BEnd}^{B4,Strt} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Sequence 5 Measurement</u>

With (21), (3) and (5) for the four sequential +180 deg rotations simplify to

$$\begin{gathered} \underline{\phi}_{End}^{BStrt} = \sum_i \left\{ C_{Bi,Strt}^{BStrt} \left[\left(\kappa_{LinScal} + \kappa_{Asym} \right) \underline{u}_i^{Bi,Strt} \pi + 2 \left(\underline{u}_i^{Bi,Strt} \times \right) \kappa_{Mis} \underline{u}_i^{Bi,Strt} \right] \right\} \\ \Delta \hat{\underline{a}}_H^{BStrt} = g\, \underline{u}_{Dwn}^{BStrt} \times \underline{\phi}_{End}^{BStrt} \end{gathered} \quad (24)$$

Substituting (22) and (23) in (24) with (6) for $\kappa_{LinScal}$, κ_{Asym}, and κ_{Mis} then obtains for the terms making up $\underline{\phi}_{End}^{BStrt}$:

$$\left(\underline{u}_1^{B1,Strt}\times\right)\kappa_{Mis}\underline{u}_1^{B1,Strt}=\left(\underline{u}_3^{B3,Strt}\times\right)\kappa_{Mis}\underline{u}_3^{B3,Strt}$$

$$=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}0&\kappa_{xy}&\kappa_{xz}\\\kappa_{yx}&0&\kappa_{yz}\\\kappa_{zx}&\kappa_{zy}&0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}0\\\kappa_{yx}\\\kappa_{zx}\end{bmatrix}=\begin{bmatrix}0\\-\kappa_{zx}\\\kappa_{yx}\end{bmatrix}$$

$$\left(\kappa_{LinScal}+\kappa_{Asym}\right)\underline{u}_1^{B1,Strt}\pi+2\left(\underline{u}_1^{B1,Strt}\times\right)\kappa_{Mis}\underline{u}_1^{B1,Strt}$$

$$=\left(\kappa_{LinScal}+\kappa_{Asym}\right)\underline{u}_3^{B1,Strt}\pi+2\left(\underline{u}_3^{B3,Strt}\times\right)\kappa_{Mis}\underline{u}_3^{B3,Strt}$$

$$=\begin{bmatrix}\kappa_{xx}+\kappa_{xxx}\\0\\0\end{bmatrix}\pi+2\begin{bmatrix}0\\-\kappa_{zx}\\\kappa_{yx}\end{bmatrix}=\begin{bmatrix}\left(\kappa_{xx}+\kappa_{xxx}\right)\pi\\-2\,\kappa_{zx}\\2\,\kappa_{yx}\end{bmatrix}$$

(25)

$$\left(\underline{u}_2^{B2,Strt}\times\right)\kappa_{Mis}\underline{u}_2^{B2,Strt}=\left(\underline{u}_4^{B4,Strt}\times\right)\kappa_{Mis}\underline{u}_4^{B4,Strt}$$

$$=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&\kappa_{xy}&\kappa_{xz}\\\kappa_{yx}&0&\kappa_{yz}\\\kappa_{zx}&\kappa_{zy}&0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\kappa_{xz}\\\kappa_{yz}\\0\end{bmatrix}=\begin{bmatrix}-\kappa_{yz}\\\kappa_{xz}\\0\end{bmatrix}$$

$$\left(\kappa_{LinScal}+\kappa_{Asym}\right)\underline{u}_2^{B2,Strt}\pi+2\left(\underline{u}_2^{B2,Strt}\times\right)\kappa_{Mis}\underline{u}_2^{B2,Strt}$$

$$=\left(\kappa_{LinScal}+\kappa_{Asym}\right)\underline{u}_4^{B4,Strt}\pi+2\left(\underline{u}_4^{B4,Strt}\times\right)\kappa_{Mis}\underline{u}_4^{B4,Strt}$$

$$=\begin{bmatrix}0\\0\\\left(\kappa_{zz}+\kappa_{zzz}\right)\end{bmatrix}\pi+2\begin{bmatrix}-\kappa_{yz}\\\kappa_{xz}\\0\end{bmatrix}=\begin{bmatrix}-2\,\kappa_{yz}\\2\,\kappa_{xz}\\\left(\kappa_{zz}+\kappa_{zzz}\right)\pi\end{bmatrix}$$

$$
\begin{aligned}
&C_{B1,Strt}^{BStrt}\left[\left(\kappa_{LinScal}+\kappa_{Asym}\right)\underline{u}_1^{B1,Strt}\pi+2\left(\underline{u}_1^{B1,Strt}\times\right)\kappa_{Mis}\underline{u}_1^{B1,Strt}\right] \\
&\quad=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\left(\kappa_{xx}+\kappa_{xxx}\right)\pi\\-2\,\kappa_{zx}\\2\,\kappa_{yx}\end{bmatrix}=\begin{bmatrix}\left(\kappa_{xx}+\kappa_{xxx}\right)\pi\\-2\,\kappa_{zx}\\2\,\kappa_{yx}\end{bmatrix} \\
&C_{B2,Strt}^{BStrt}\left[\left(\kappa_{LinScal}+\kappa_{Asym}\right)\underline{u}_2^{B2,Strt}\pi+2\left(\underline{u}_2^{B2,Strt}\times\right)\kappa_{Mis}\underline{u}_2^{B2,Strt}\right] \\
&\quad=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}-2\,\kappa_{yz}\\2\,\kappa_{xz}\\\left(\kappa_{zz}+\kappa_{zzz}\right)\pi\end{bmatrix}=\begin{bmatrix}-2\,\kappa_{yz}\\-2\,\kappa_{xz}\\-\left(\kappa_{zz}+\kappa_{zzz}\right)\pi\end{bmatrix} \\
&C_{B3,Strt}^{BStrt}\left[\left(\kappa_{LinScal}+\kappa_{Asym}\right)\underline{u}_3^{B3,Strt}\pi+2\left(\underline{u}_3^{B3,Strt}\times\right)\kappa_{Mis}\underline{u}_3^{B3,Strt}\right] \\
&\quad=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}\left(\kappa_{xx}+\kappa_{xxx}\right)\pi\\-2\,\kappa_{zx}\\2\,\kappa_{yx}\end{bmatrix}=\begin{bmatrix}-\left(\kappa_{xx}+\kappa_{xxx}\right)\pi\\-2\,\kappa_{zx}\\-2\,\kappa_{yx}\end{bmatrix} \\
&C_{B4,Strt}^{BStrt}\left[\left(\kappa_{LinScal}+\kappa_{Asym}\right)\underline{u}_4^{B4,Strt}\pi+2\left(\underline{u}_4^{B4,Strt}\times\right)\kappa_{Mis}\underline{u}_4^{B4,Strt}\right] \\
&\quad=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-2\,\kappa_{yz}\\2\,\kappa_{xz}\\\left(\kappa_{zz}+\kappa_{zzz}\right)\pi\end{bmatrix}=\begin{bmatrix}2\,\kappa_{yz}\\-2\,\kappa_{xz}\\\left(\kappa_{zz}+\kappa_{zzz}\right)\pi\end{bmatrix}
\end{aligned}
\tag{26}
$$

Substituting (26) in (24) and applying (10) for a MARS type B frame, then finds for $\underline{\phi}_{End}^{BStrt}$:

$$
\begin{aligned}
\underline{\phi}_{End}^{BStrt}&=\sum_i\left\{\begin{bmatrix}\left(\kappa_{xx}+\kappa_{xxx}\right)\pi\\-2\,\kappa_{zx}\\2\,\kappa_{yx}\end{bmatrix}+\begin{bmatrix}-2\,\kappa_{yz}\\-2\,\kappa_{xz}\\-\left(\kappa_{zz}+\kappa_{zzz}\right)\pi\end{bmatrix}+\begin{bmatrix}-\left(\kappa_{xx}+\kappa_{xxx}\right)\pi\\-2\,\kappa_{zx}\\-2\,\kappa_{yx}\end{bmatrix}+\begin{bmatrix}2\,\kappa_{yz}\\-2\,\kappa_{xz}\\\left(\kappa_{zz}+\kappa_{zzz}\right)\pi\end{bmatrix}\right\} \\
&=\sum_i\left\{\begin{bmatrix}\left(\kappa_{xx}+\kappa_{xxx}\right)\pi\\-\upsilon_{zx}\\\upsilon_{xy}\end{bmatrix}+\begin{bmatrix}-\upsilon_{yz}\\-\upsilon_{zx}\\-\left(\kappa_{zz}+\kappa_{zzz}\right)\pi\end{bmatrix}+\begin{bmatrix}-\left(\kappa_{xx}+\kappa_{xxx}\right)\pi\\-\upsilon_{zx}\\-\upsilon_{xy}\end{bmatrix}+\begin{bmatrix}\upsilon_{yz}\\-\upsilon_{zx}\\\left(\kappa_{zz}+\kappa_{zzz}\right)\pi\end{bmatrix}\right\}=\begin{bmatrix}0\\-4\,\upsilon_{zx}\\0\end{bmatrix}
\end{aligned}
\tag{27}
$$

With (27) and (22) for $\underline{u}_{Dwn}^{BStrt}$, (24) obtains for $\Delta\hat{\underline{a}}_H^{BStrt}$:

$$\Delta \underline{\hat{a}}_H^{B_{Strt}} = g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -4\, \upsilon_{zx} \\ 0 \end{bmatrix} = \begin{bmatrix} 4\, g\, \upsilon_{zx} \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

The x component in (28) is identical to that for sequence 5 in Part 1 [3, Eqs. (16)].

Sequence 5 Unmodelled Gyro Bias Error Effect

Table 1 rotation sequence 6 is used to determine the orthogonality error between the x and y gyros. For the Table 1 sequence 5 four +180 deg rotations, (12) and (13) with (15) for $C_{B_{End}}^{B_{Strt}}$ simplify to

$$\begin{aligned} \Delta \underline{\phi}_{GyroBiasRot}^{B_{Strt}} &= \sum_i C_{B_{i,Strt}}^{B_{Strt}} \left[I + \frac{2}{\pi} \left(\underline{u}_i^{B_{i,Strt}} \times \right) + \left(\underline{u}_i^{B_{i,Strt}} \times \right)^2 \right] \frac{\pi}{\dot{\beta}}\, \underline{\kappa}_{Bias} \\ e\left(\Delta \underline{\hat{a}}^{B_{Strt}} \right)_{GyroBias} &= g\, \underline{u}_{Dwn}^{B_{Strt}} \times \left(\Delta \underline{\phi}_{GyroBias_{Rot}}^{B_{Strt}} + T_{Meas}\, \underline{\kappa}_{Bias} \right) \end{aligned} \tag{29}$$

Substituting (22) and (23) in (29) then obtains for terms in $\Delta \underline{\phi}_{GyroBiasRot}^{B_{Strt}}$:

$$\begin{aligned} & I + \frac{2}{\pi} \left(\underline{u}_1^{B_{1,Strt}} \times \right) + \left(\underline{u}_1^{B_{1,Strt}} \times \right)^2 = I + \frac{2}{\pi} \left(\underline{u}_3^{B_{3,Strt}} \times \right) + \left(\underline{u}_3^{B_{3,Strt}} \times \right)^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{2}{\pi} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \frac{2}{\pi} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ & I + \frac{2}{\pi} \left(\underline{u}_2^{B_{2,Strt}} \times \right) + \left(\underline{u}_2^{B_{2,Strt}} \times \right)^2 = I + \frac{2}{\pi} \left(\underline{u}_4^{B_{4,Strt}} \times \right) + \left(\underline{u}_4^{B_{4,Strt}} \times \right)^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{2}{\pi} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{2}{\pi} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{30}$$

$$C_{B1,Strt}^{BStrt}\left[I+\frac{2}{\pi}\left(\underline{u}_1^{B1,Strt}\times\right)+\left(\underline{u}_1^{B1,Strt}\times\right)^2\right]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\left\{\frac{2}{\pi}\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\right\}$$

$$=\frac{2}{\pi}\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$C_{B2,Strt}^{BStrt}\left[I+\frac{2}{\pi}\left(\underline{u}_2^{B2,Strt}\times\right)+\left(\underline{u}_2^{B2,Strt}\times\right)^2\right]=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\left\{\frac{2}{\pi}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$$

$$=\frac{2}{\pi}\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&-1\end{bmatrix}$$

(31)

$$C_{B3,Strt}^{BStrt}\left[I+\frac{2}{\pi}\left(\underline{u}_3^{B3,Strt}\times\right)+\left(\underline{u}_3^{B3,Strt}\times\right)^2\right]=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\left\{\frac{2}{\pi}\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\right\}$$

$$=\frac{2}{\pi}\begin{bmatrix}0&0&0\\0&0&-1\\0&-1&0\end{bmatrix}+\begin{bmatrix}-1&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

$$C_{B4,Strt}^{BStrt}\left[I+\frac{2}{\pi}\left(\underline{u}_4^{B4,Strt}\times\right)+\left(\underline{u}_4^{B4,Strt}\times\right)^2\right]=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\left\{\frac{2}{\pi}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$$

$$=\frac{2}{\pi}\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

$$\sum_i C_{Bi,Strt}^{BStrt}\left[I+\frac{2}{\pi}\left(\underline{u}_i^{Bi,Strt}\times\right)+\left(\underline{u}_i^{Bi,Strt}\times\right)^2\right]$$

$$=\frac{2}{\pi}\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+\frac{2}{\pi}\begin{bmatrix}0&0&0\\0&0&-1\\0&-1&0\end{bmatrix}+\begin{bmatrix}-1&0&0\\0&0&0\\0&0&0\end{bmatrix} \tag{32}$$

$$+\frac{2}{\pi}\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&-1\end{bmatrix}+\frac{2}{\pi}\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}=\frac{4}{\pi}\begin{bmatrix}0&0&0\\-1&0&-1\\0&0&0\end{bmatrix}$$

Substituting (32) in (29) with (6) for $\underline{\kappa}_{Bias}$ gives for $\Delta\underline{\phi}_{GyroBiasRot}^{BStrt}$:

$$\Delta\underline{\phi}_{GyroBiasRot}^{BStrt} = \sum_i C_{B_{i,Strt}}^{BStrt} \left[I + \frac{2}{\pi}\left(\underline{u}_i^{B_{i,Strt}} \times\right) + \left(\underline{u}_i^{B_{i,Strt}} \times\right)^2 \right] \frac{\pi}{\dot{\beta}} \underline{\kappa}_{Bias}$$
$$= \frac{4}{\dot{\beta}} \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_z \end{bmatrix} = \begin{bmatrix} 0 \\ -4(\kappa_x + \kappa_z)/\dot{\beta} \\ 0 \end{bmatrix} \tag{33}$$

With (33) for $\Delta\underline{\phi}_{GyroBias_{Rot}}^{BStrt}$ and (22) for $\underline{u}_{Dwn}^{BStrt}$, $e\left(\Delta\hat{\underline{a}}^{BStrt}\right)_{GyroBias}$ in (29) then becomes:

$$e\left(\Delta\hat{\underline{a}}^{BStrt}\right)_{GyroBias} = g\, \underline{u}_{Dwn}^{BStrt} \times \left(\Delta\underline{\phi}_{GyroBias_{Rot}}^{BStrt} + T_{Meas}\, \underline{\kappa}_{Bias}\right)$$
$$= g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left\{ \begin{bmatrix} 0 \\ -4(\kappa_x + \kappa_z)/\dot{\beta} \\ 0 \end{bmatrix} + T_{Meas} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_z \end{bmatrix} \right\} = g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} T_{Meas}\, \kappa_x \\ -4(\kappa_x + \kappa_z)/\dot{\beta} + T_{Meas}\, \kappa_y \\ T_{Meas}\, \kappa_z \end{bmatrix} \tag{34}$$
$$= \begin{bmatrix} 4g\left[(\kappa_x + \kappa_z)/\dot{\beta} - T_{Meas}\, \kappa_y / 4\right] \\ g\, T_{Meas}\, \kappa_x \\ 0 \end{bmatrix}$$

ROTATION SEQUENCE 6

From Table 1, sequence 6 consists of eight interlaced +180 degree and +90 deg rotations (four at 180 deg, four at 90 deg) which return the IMU to its starting orientation. Then

$$C_{B_{End}}^{BStrt} = I \qquad \delta\hat{\underline{a}}_{SF_{End}}^{BEnd} = \delta\hat{\underline{a}}_{SF_{Strt}}^{BStrt} \tag{35}$$

Table 1 sequence 5 also shows that IMU axis x is initially down and the sequence of eight IMU axis rotations is +180 y, +90 z, +180 x, +90 z, +180 y, +90 z, +180 x, +90 z in that order:

$$\underline{u}_{Dwn}^{B_{Strt}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad +180:\ \underline{u}_1^{B1,Strt} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad +90:\ \underline{u}_2^{B2,Strt} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad +180:\ \underline{u}_3^{B3,Strt} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$+90:\ \underline{u}_4^{B4,Strt} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad +180:\ \underline{u}_5^{B5,Strt} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad +90:\ \underline{u}_6^{B6,Strt} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{36}$$

$$+180:\ \underline{u}_7^{B7,Strt} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad +90:\ \underline{u}_8^{B8,Strt} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

From (36) and (3) we can then write:

$$C_{B1,Strt}^{BStrt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

+180 about y: $$C_{B2,Strt}^{BStrt} = C_{B1,Strt}^{BStrt}\, C_{B2,Strt}^{B1,Strt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

+90 about z: $$C_{B3,Strt}^{BStrt} = C_{B2,Strt}^{BStrt}\, C_{B3,Strt}^{B2,Strt} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

+180 about x: $$C_{B4,Strt}^{BStrt} = C_{B3,Strt}^{BStrt}\, C_{B4,Strt}^{B3,Strt} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

+90 about z: $$C_{B5,Strt}^{BStrt} = C_{B4,Strt}^{BStrt}\, C_{B5,Strt}^{B4,Strt} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (37)$$

+180 about y: $$C_{B6,Strt}^{BStrt} = C_{B5,Strt}^{BStrt}\, C_{B6,Strt}^{B5,Strt} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

+90 about z: $$C_{B7,Strt}^{BStrt} = C_{B6,Strt}^{BStrt}\, C_{B7,Strt}^{B6,Strt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

+180 about x: $$C_{B8,Strt}^{BStrt} = C_{B7,Strt}^{BStrt}\, C_{B8,Strt}^{B7,Strt} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

+90 about z: $$C_{BEnd}^{BStrt} = C_{B8,Strt}^{BStrt}\, C_{BEnd}^{B8,Strt} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Sequence 6 Measurement</u>

With (35), (3) and (5) for the four +180 deg and four +90 deg rotations become

$$\underline{\phi}_{End}^{BStrt} = \sum_{i=1,3,5,7} \left\{ C_{B_{i,Strt}}^{BStrt} \left[\left(\kappa_{LinScal} + \kappa_{Asym} \right) \underline{u}_i^{B_{i,Strt}} \pi + 2 \left(\underline{u}_i^{B_{i,Strt}} \times \right) \kappa_{Mis} \underline{u}_i^{B_{i,Strt}} \right] \right\}$$

$$+ \sum_{i=2,4,6,8} \left\{ C_{B_{i,Strt}}^{BStrt} \left[\left(\kappa_{LinScal} + \kappa_{Asym} \right) \underline{u}_i^{B_{i,Strt}} \frac{\pi}{2} + \left[I + \left(\underline{u}_i^{B_{i,Strt}} \times \right) \right] \kappa_{Mis} \underline{u}_i^{B_{i,Strt}} \right] \right\} \quad (38)$$

$$\Delta \hat{\underline{a}}_H^{BStrt} = g \, \underline{u}_{Dwn}^{BStrt} \times \underline{\phi}_{End}^{BStrt}$$

Substituting (36) and (37) in (38) with (6) for $\kappa_{LinScal}$, κ_{Asym}, κ_{Mis} then obtains for terms making up $\underline{\phi}_{End}^{BStrt}$:

For the IMU 180 deg y axis rotations: $i = 1, 5$ and $\underline{u}_i^{B_{i,Strt}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\left(\kappa_{LinScal} + \kappa_{Asym} \right) \underline{u}_i^{B_{i,Strt}} \pi + 2 \left(\underline{u}_i^{B_{i,Str}} \times \right) \kappa_{Mis} \underline{u}_i^{B_{i,Str}}$$

$$= \begin{bmatrix} \kappa_{xx} + \kappa_{xxx} \\ \kappa_{yy} + \kappa_{yyy} \\ \kappa_{zz} + \kappa_{zzz} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \pi + 2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \kappa_{xy} & \kappa_{xz} \\ \kappa_{yx} & 0 & \kappa_{yz} \\ \kappa_{zx} & \kappa_{zy} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \left(\kappa_{yy} + \kappa_{yyy} \right) \pi \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \kappa_{xy} \\ 0 \\ \kappa_{zy} \end{bmatrix} = \begin{bmatrix} 2 \, \kappa_{zy} \\ \left(\kappa_{yy} + \kappa_{yyy} \right) \pi \\ -2 \, \kappa_{xy} \end{bmatrix}$$

$$(39)$$

$$C_{B_{1,Strt}}^{BStrt} \left[\left(\kappa_{LinScal} + \kappa_{Asym} \right) \underline{u}_1^{B_{1,Strt}} \pi + 2 \left(\underline{u}_1^{B_{1,Strt}} \times \right) \kappa_{Mis} \underline{u}_1^{B_{1,Strt}} \right]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \, \kappa_{zy} \\ \left(\kappa_{yy} + \kappa_{yyy} \right) \pi \\ -2 \, \kappa_{xy} \end{bmatrix} = \begin{bmatrix} 2 \, \kappa_{zy} \\ \left(\kappa_{yy} + \kappa_{yyy} \right) \pi \\ -2 \, \kappa_{xy} \end{bmatrix}$$

$$C_{B_{5,Strt}}^{BStrt} \left[\left(\kappa_{LinScal} + \kappa_{Asym} \right) \underline{u}_5^{B_{5,Strt}} \pi + 2 \left(\underline{u}_5^{B_{5,Strt}} \times \right) \kappa_{Mis} \underline{u}_5^{B_{5,Strt}} \right]$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \, \kappa_{zy} \\ \left(\kappa_{yy} + \kappa_{yyy} \right) \pi \\ -2 \, \kappa_{xy} \end{bmatrix} = \begin{bmatrix} -2 \, \kappa_{zy} \\ -\left(\kappa_{yy} + \kappa_{yyy} \right) \pi \\ -2 \, \kappa_{xy} \end{bmatrix}$$

For the IMU 180 deg x axis rotations: $i = 3,\ 7$ and $\underline{u}_i^{B_{i,Strt}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$
\begin{aligned}
&\left(\kappa_{LinScal} + \kappa_{Asym}\right)\underline{u}^{B_{i,Strt}}\pi + 2\left(\underline{u}_i^{B_{i,Strt}} \times\right)\kappa_{Mis}\underline{u}_i^{B_{i,Strt}} \\
&= \begin{bmatrix} \kappa_{xx} + \kappa_{xxx} \\ \kappa_{yy} + \kappa_{yyy} \\ \kappa_{zz} + \kappa_{zzz} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\pi + 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \kappa_{xy} & \kappa_{xz} \\ \kappa_{yx} & 0 & \kappa_{yz} \\ \kappa_{zx} & \kappa_{zy} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} \left(\kappa_{xx} + \kappa_{xxx}\right)\pi \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ -0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \kappa_{yx} \\ \kappa_{zx} \end{bmatrix} = \begin{bmatrix} \left(\kappa_{xx} + \kappa_{xxx}\right)\pi \\ -2\ \kappa_{zx} \\ 2\ \kappa_{yx} \end{bmatrix} \\
&C_{B_{3,Strt}}^{B_{Strt}}\left[\left(\kappa_{LinScal} + \kappa_{Asym}\right)\underline{u}_3^{B_{3,Strt}}\pi + 2\left(\underline{u}_3^{B_{3,Strt}} \times\right)\kappa_{Mis}\underline{u}_3^{B_{3,Strt}}\right] \\
&\qquad = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \left(\kappa_{xx} + \kappa_{xxx}\right)\pi \\ -2\ \kappa_{zx} \\ 2\ \kappa_{yx} \end{bmatrix} = \begin{bmatrix} -2\ \kappa_{zx} \\ \left(\kappa_{xx} + \kappa_{xxx}\right)\pi \\ -2\ \kappa_{yx} \end{bmatrix} \\
&C_{B_{7,Strt}}^{B_{Strt}}\left[\left(\kappa_{LinScal} + \kappa_{Asym}\right)\underline{u}_7^{B_{7,Strt}}\pi + 2\left(\underline{u}_7^{B_{7,Strt}} \times\right)\kappa_{Mis}\underline{u}_7^{B_{7,Strt}}\right] \\
&\qquad = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} \left(\kappa_{xx} + \kappa_{xxx}\right)\pi \\ -2\ \kappa_{zx} \\ 2\ \kappa_{yx} \end{bmatrix} = \begin{bmatrix} 2\ \kappa_{zx} \\ -\left(\kappa_{xx} + \kappa_{xxx}\right)\pi \\ -2\ \kappa_{yx} \end{bmatrix}
\end{aligned}
\tag{40}
$$

For the IMU 90 deg z axis rotations: $i = 2, 4, 6, 8$ and $\underline{u}_i^{Bi,Strt} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\left[I + \left(\underline{u}_i^{Bi,Strt} \times \right) \right] \kappa_{Mis} \underline{u}_i^{Bi,Strt} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \kappa_{xy} & \kappa_{xz} \\ \kappa_{yx} & 0 & \kappa_{yz} \\ \kappa_{zx} & \kappa_{zy} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \kappa_{xz} \\ \kappa_{yz} \\ 0 \end{bmatrix} = \begin{bmatrix} \kappa_{xz} - \kappa_{yz} \\ \kappa_{xz} + \kappa_{yz} \\ 0 \end{bmatrix}$$

$$\left(\kappa_{LinScal} + \kappa_{Asym} \right) \underline{u}_i^{Bi,Strt} \frac{\pi}{2} + \left[I + \left(\underline{u}_i^{Bi,Strt} \times \right) \right] \kappa_{Mis} \underline{u}_i^{Bi,Strt}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \left(\kappa_{zz} + \kappa_{zzz} \right) \pi / 2 \end{bmatrix} + \begin{bmatrix} \kappa_{xz} - \kappa_{yz} \\ \kappa_{xz} + \kappa_{yz} \\ 0 \end{bmatrix} = \begin{bmatrix} \kappa_{xz} - \kappa_{yz} \\ \kappa_{xz} + \kappa_{yz} \\ \left(\kappa_{zz} + \kappa_{zzz} \right) \pi / 2 \end{bmatrix}$$

$$C_{B2,Strt}^{BStrt} \left\{ \left(\kappa_{LinScal} + \kappa_{Asym} \right) \underline{u}_2^{B2,Strt} \frac{\pi}{2} + \left[I + \left(\underline{u}_2^{B2,Strt} \times \right) \right] \kappa_{Mis} \underline{u}_2^{B2,Strt} \right\}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \kappa_{xz} - \kappa_{yz} \\ \kappa_{xz} + \kappa_{yz} \\ \kappa_{zz} \, \pi / 2 \end{bmatrix} = \begin{bmatrix} -\left(\kappa_{xz} - \kappa_{yz} \right) \\ \kappa_{xz} + \kappa_{yz} \\ -\left(\kappa_{zz} + \kappa_{zzz} \right) \pi / 2 \end{bmatrix} \qquad (41)$$

$$C_{B4,Strt}^{BStrt} \left\{ \left(\kappa_{LinScal} + \kappa_{Asym} \right) \underline{u}_4^{B4,Strt} \frac{\pi}{2} + \left[I + \left(\underline{u}_4^{B4,Strt} \times \right) \right] \kappa_{Mis} \underline{u}_4^{B4,Strt} \right\}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \kappa_{xz} - \kappa_{yz} \\ \kappa_{xz} + \kappa_{yz} \\ \left(\kappa_{zz} + \kappa_{zzz} \right) \pi / 2 \end{bmatrix} = \begin{bmatrix} -\left(\kappa_{xz} + \kappa_{yz} \right) \\ \kappa_{xz} - \kappa_{yz} \\ \left(\kappa_{zz} + \kappa_{zzz} \right) \pi / 2 \end{bmatrix}$$

$$C_{B6,Strt}^{BStrt} \left\{ \left(\kappa_{LinScal} + \kappa_{Asym} \right) \underline{u}_6^{B6,Strt} \frac{\pi}{2} + \left[I + \left(\underline{u}_6^{B6,Strt} \times \right) \right] \kappa_{Mis} \underline{u}_6^{B6,Strt} \right\}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \kappa_{xz} - \kappa_{yz} \\ \kappa_{xz} + \kappa_{yz} \\ \left(\kappa_{zz} + \kappa_{zzz} \right) \pi / 2 \end{bmatrix} = \begin{bmatrix} \kappa_{xz} - \kappa_{yz} \\ -\left(\kappa_{xz} + \kappa_{yz} \right) \\ -\left(\kappa_{zz} + \kappa_{zzz} \right) \pi / 2 \end{bmatrix}$$

$$C_{B8,Strt}^{BStrt} \left\{ \left(\kappa_{LinScal} + \kappa_{Asym} \right) \underline{u}_8^{B8,Strt} \frac{\pi}{2} + \left[I + \left(\underline{u}_8^{B8,Strt} \times \right) \right] \kappa_{Mis} \underline{u}_8^{B8,Strt} \right\}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \kappa_{xz} - \kappa_{yz} \\ \kappa_{xz} + \kappa_{yz} \\ \left(\kappa_{zz} + \kappa_{zzz} \right) \pi / 2 \end{bmatrix} = \begin{bmatrix} \kappa_{xz} + \kappa_{yz} \\ -\left(\kappa_{xz} - \kappa_{yz} \right) \\ \left(\kappa_{zz} + \kappa_{zzz} \right) \pi / 2 \end{bmatrix}$$

Substituting (39) – (41) into (38) and applying (10) for a MARS type B frame then finds for $\underline{\phi}_{End}^{BStrt}$:

$$\underline{\phi}_{End}^{BStrt} = \sum_i \left\{ \begin{array}{l} \begin{bmatrix} 2\,\kappa_{zy} \\ (\kappa_{yy}+\kappa_{yyy})\pi \\ -2\,\kappa_{xy} \end{bmatrix} + \begin{bmatrix} -(\kappa_{xz}-\kappa_{yz}) \\ \kappa_{xz}+\kappa_{yz} \\ -(\kappa_{zz}+\kappa_{zzz})\,\pi/2 \end{bmatrix} + \begin{bmatrix} -2\,\kappa_{zx} \\ (\kappa_{xx}+\kappa_{xxx})\pi \\ -2\,\kappa_{yx} \end{bmatrix} + \begin{bmatrix} -(\kappa_{xz}+\kappa_{yz}) \\ \kappa_{xz}-\kappa_{yz} \\ (\kappa_{zz}+\kappa_{zzz})\,\pi/2 \end{bmatrix} \\ + \begin{bmatrix} -2\,\kappa_{zy} \\ -(\kappa_{yy}+\kappa_{yyy})\pi \\ -2\,\kappa_{xy} \end{bmatrix} + \begin{bmatrix} \kappa_{xz}-\kappa_{yz} \\ -(\kappa_{xz}+\kappa_{yz}) \\ -(\kappa_{zz}+\kappa_{zzz})\,\pi/2 \end{bmatrix} + \begin{bmatrix} 2\,\kappa_{zx} \\ -(\kappa_{xx}+\kappa_{xxx})\pi \\ -2\,\kappa_{yx} \end{bmatrix} + \begin{bmatrix} \kappa_{xz}+\kappa_{yz} \\ -(\kappa_{xz}-\kappa_{yz}) \\ (\kappa_{zz}+\kappa_{zzz})\,\pi/2 \end{bmatrix} \end{array} \right\} \quad (42)$$

$$= \begin{bmatrix} 0 \\ 0 \\ -4(\kappa_{xy}+\kappa_{yx}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -4\,\upsilon_{xy} \end{bmatrix}$$

With (42) for $\underline{\phi}_{End}^{BStrt}$ and (36) for $\underline{u}_{Dwn}^{BStrt}$, (38) obtains for $\Delta\hat{\underline{a}}_H^{BStrt}$:

$$\Delta\hat{\underline{a}}_H^{BStrt} = g \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -4\,\upsilon_{xy} \end{bmatrix} = \begin{bmatrix} 0 \\ 4\,g\,\upsilon_{xy} \\ 0 \end{bmatrix} \quad (43)$$

The y component in (43) is identical to that for sequence 6 in Part 1 [3, Eqs. (16)].

Sequence 6 Unmodelled Gyro Bias Error Effect

For the Table 1 sequence 6 four +180 deg and four 90 deg rotations, (12) and (13) with (35) for $C_{B_{End}}^{BStrt}$ simplify to

$$\begin{aligned} \Delta\underline{\phi}_{GyroBiasRot}^{BStrt} &= \sum_{i=1,3,5,7} \left\{ C_{B_{i,Strt}}^{BStrt} \left[I + \frac{2}{\pi}\left(\underline{u}_i^{Bi}\times\right) + \left(\underline{u}_i^{Bi}\times\right)^2 \right] \frac{\pi}{\dot{\beta}}\, \underline{\kappa}_{Bias} \right\} \\ &+ \sum_{i=2,4,6,8} \left\{ C_{B_{i,Strt}}^{BStrt} \left[I + \frac{2}{\pi}\left(\underline{u}_i^{Bi}\times\right) + \left(1-\frac{2}{\pi}\right)\left(\underline{u}_i^{Bi}\times\right)^2 \right] \frac{\pi/2}{\dot{\beta}}\, \underline{\kappa}_{Bias} \right\} \quad (44) \\ e\left(\Delta\hat{\underline{a}}^{BStrt}\right)_{GyroBias} &= g\, \underline{u}_{Dwn}^{BStrt} \times \left(\Delta\underline{\phi}_{GyroBias_{Rot}}^{BStrt} + T_{Meas}\, \underline{\kappa}_{Bias} \right) \end{aligned}$$

Substituting (36) and (37) in (44) then obtains for terms in $\Delta\underline{\phi}_{GyroBiasRot}^{BStrt}$:

$$
\begin{gathered}
I+\frac{2}{\pi}\left(\underline{u}_1^{B1,Strt}\times\right)+\left(\underline{u}_1^{B1,Strt}\times\right)^2=I+\frac{2}{\pi}\left(\underline{u}_5^{B3,Strt}\times\right)+\left(\underline{u}_5^{B3,Strt}\times\right)^2 \\
=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\frac{2}{\pi}\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}+\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}=\frac{2}{\pi}\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} \\[2ex]
I+\frac{2}{\pi}\left(\underline{u}_3^{B3,Strt}\times\right)+\left(\underline{u}_3^{B3,Strt}\times\right)^2=I+\frac{2}{\pi}\left(\underline{u}_7^{B7,Strt}\times\right)+\left(\underline{u}_7^{B7,Strt}\times\right)^2 \\
=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\frac{2}{\pi}\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}=\frac{2}{\pi}\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} \\[2ex]
\left[I+\frac{2}{\pi}\left(\underline{u}_2^{B2,Strt}\times\right)+\left(1-\frac{2}{\pi}\right)\left(\underline{u}_2^{B2,Strt}\times\right)^2\right]=\left[I+\frac{2}{\pi}\left(\underline{u}_4^{B4,Strt}\times\right)+\left(1-\frac{2}{\pi}\right)\left(\underline{u}_4^{B4,Strt}\times\right)^2\right] \\
=\left[I+\frac{2}{\pi}\left(\underline{u}_6^{B6,Strt}\times\right)+\left(1-\frac{2}{\pi}\right)\left(\underline{u}_6^{B6,Strt}\times\right)^2\right]=\left[I+\frac{2}{\pi}\left(\underline{u}_8^{B8,Strt}\times\right)+\left(1-\frac{2}{\pi}\right)\left(\underline{u}_8^{B8,Strt}\times\right)^2\right] \\
=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\frac{2}{\pi}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}+\left(1-\frac{2}{\pi}\right)\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}=\frac{2}{\pi}\begin{bmatrix}1&-1&0\\1&1&0\\0&0&1\end{bmatrix}
\end{gathered}
\tag{45}
$$

$$
\begin{aligned}
&\sum_{i=1,5}\left\{C_{B_{i,Strt}}^{BStrt}\left[I+\frac{2}{\pi}\left(\underline{u}_i^{Bi}\times\right)+\left(\underline{u}_i^{Bi}\times\right)^2\right]\right\}+\sum_{i=3,7}\left\{C_{B_{i,Strt}}^{BStrt}\left[I+\frac{2}{\pi}\left(\underline{u}_i^{Bi}\times\right)+\left(\underline{u}_i^{Bi}\times\right)^2\right]\right\} \\
&\quad+\sum_{i=2,4,6,8}\left\{C_{B_{i,Strt}}^{BStrt}\left[I+\frac{2}{\pi}\left(\underline{u}_i^{Bi}\times\right)+\left(1-\frac{2}{\pi}\right)\left(\underline{u}_i^{Bi}\times\right)^2\right]\right\} \\
&=\left\{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\right\}\left\{\frac{2}{\pi}\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\right\} \\
&\quad+\left\{\begin{bmatrix}0&1&0\\1&0&0\\0&0&-1\end{bmatrix}+\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&-1\end{bmatrix}\right\}\left\{\frac{2}{\pi}\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\right\} \\
&\quad+\left\{\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}+\begin{bmatrix}0&-1&0\\1&0&0\\0&0&1\end{bmatrix}+\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}+\begin{bmatrix}0&1&0\\-1&0&0\\0&0&1\end{bmatrix}\right\}\frac{2}{\pi}\begin{bmatrix}1&-1&0\\1&1&0\\0&0&1\end{bmatrix} \\
&=\begin{bmatrix}0&0&0\\0&0&0\\0&0&2\end{bmatrix}\left\{\frac{2}{\pi}\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}\right\}+\begin{bmatrix}0&0&0\\0&0&0\\0&0&-2\end{bmatrix}\left\{\frac{2}{\pi}\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}+\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\right\} \\
&\quad+\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\frac{2}{\pi}\begin{bmatrix}1&-1&0\\1&1&0\\0&0&1\end{bmatrix} \\
&=\frac{2}{\pi}\begin{bmatrix}0&0&0\\0&0&0\\-2&0&0\end{bmatrix}+\frac{2}{\pi}\begin{bmatrix}0&0&0\\0&0&0\\0&-2&0\end{bmatrix}=-\frac{4}{\pi}\begin{bmatrix}0&0&0\\0&0&0\\1&1&0\end{bmatrix}
\end{aligned}
\tag{46}
$$

Substituting (46) in (44) with (6) for $\underline{\kappa}_{Bias}$ gives for $\Delta\underline{\phi}_{GyroBiasRot}^{BStrt}$:

$$
\begin{aligned}
\Delta\underline{\phi}_{GyroBiasRot}^{BStrt}&=\sum_i C_{B_{i,Strt}}^{BStrt}\left[I+\frac{2}{\pi}\left(\underline{u}_i^{B_{i,Strt}}\times\right)+\left(\underline{u}_i^{B_{i,Strt}}\times\right)^2\right]\frac{\pi}{\dot{\beta}}\,\underline{\kappa}_{Bias} \\
&=\frac{4}{\dot{\beta}}\begin{bmatrix}0&0&0\\-1&0&-1\\0&0&0\end{bmatrix}\begin{bmatrix}\kappa_x\\\kappa_y\\\kappa_z\end{bmatrix}=\begin{bmatrix}0\\-4\left(\kappa_x+\kappa_z\right)/\,\dot{\beta}\\0\end{bmatrix}
\end{aligned}
\tag{47}
$$

With (47) for $\Delta\underline{\phi}_{GyroBiasRot}^{BStrt}$ and (36) for $\underline{u}_{Dwn}^{BStrt}$, $e\left(\Delta\hat{\underline{a}}^{BStrt}\right)_{GyroBias}$ in (44) then becomes:

$$e\left(\Delta\hat{\underline{a}}^{BStrt}\right)_{GyroBias} = g\ \underline{u}_{Dwn}^{BStrt} \times \left(\Delta\underline{\phi}_{GyroBias_{Rot}}^{BStrt} + T_{Meas}\ \underline{\kappa}_{Bias}\right)$$

$$= g\begin{bmatrix}1\\0\\0\end{bmatrix} \times \left(-\frac{4}{\dot{\beta}}\begin{bmatrix}0\\0\\\kappa_x+\kappa_y\end{bmatrix} + T_{Meas}\begin{bmatrix}\kappa_x\\\kappa_y\\\kappa_z\end{bmatrix}\right) = g\begin{bmatrix}1\\0\\0\end{bmatrix} \times \begin{bmatrix}T_{Meas}\ \kappa_x\\T_{Meas}\ \kappa_y\\-4\left(\kappa_x+\kappa_y\right)/\dot{\beta} + T_{Meas}\ \kappa_z\end{bmatrix} \tag{48}$$

$$= \begin{bmatrix}0\\4g\left[\left(\kappa_x+\kappa_y\right)/\dot{\beta} - T_{Meas}\ \kappa_z/4\right]\\g\,T_{Meas}\ \kappa_y\end{bmatrix}$$

ROTATION SEQUENCE 7

Rotation sequence 7 is representative of sequences 7 – 12 used to determine the misalignment errors between the accelerometers and gyros. In addition, sequence 7 is used to determine the z accelerometer bias error. From Table 1, sequence 7 consists of a single +180 degree rotation around IMU axis x from an initial IMU y axis down orientation:

$$\underline{u}_{Dwn}^{BStrt} = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \underline{u}_1^{B1,Strt} = \begin{bmatrix}1\\0\\0\end{bmatrix} \tag{49}$$

Then from (3) and (5):

$$C_{B1,Strt}^{BStrt} = I \qquad C_{BEnd}^{BStrt} = C_{B1,Strt}^{BStrt}\ C_{B2,Strt}^{B1,Strt} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$
$$\underline{u}_{Dwn}^{BEnd} = \left(C_{BEnd}^{BStrt}\right)^T \underline{u}_{Dwn}^{BStrt} = \begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\\0\end{bmatrix} \tag{50}$$

From (49), (50) and the definition of $A_{SFSign}^{BStrt}, A_{SFSign}^{BEnd}$ as diagonal matrices having elements equal to the negative of $\underline{u}_{Dwn}^{BStrt}$, $\underline{u}_{Dwn}^{BEnd}$:

$$A_{SFSign}^{BStrt} = \begin{bmatrix}0&0&0\\0&-1&0\\0&0&0\end{bmatrix} \qquad A_{SFSign}^{BEnd} = \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} \tag{51}$$

Sequence 7 Measurement

With (49) – (50) and (6) for $\kappa_{LinScal}$, κ_{Asym}, κ_{Mis}, (3) for $\underline{\phi}_{End}^{B_{Strt}}$ becomes for the single +180 deg rotation:

$$
\underline{\phi}_{End}^{B_{Strt}} = \left(\kappa_{LinScal} + \kappa_{Asym}\right) \underline{u}_1^{B_{1,Strt}} \pi + 2\left(\underline{u}_1^{B_{1,Strt}} \times\right) \kappa_{Mis} \underline{u}_1^{B_{1,Strt}}
$$

$$
= \begin{bmatrix} \kappa_{xx} + \kappa_{xxx} & 0 & 0 \\ 0 & \kappa_{yy} + \kappa_{yyy} & 0 \\ 0 & 0 & \kappa_{zz} + \kappa_{zzz} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \pi + 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \kappa_{xy} & \kappa_{xz} \\ \kappa_{yx} & 0 & \kappa_{yz} \\ \kappa_{zx} & \kappa_{zy} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (52)
$$

$$
= \begin{bmatrix} \kappa_{xx} + \kappa_{xxx} \\ 0 \\ 0 \end{bmatrix} \pi + 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \kappa_{yx} \\ \kappa_{zx} \end{bmatrix} = \begin{bmatrix} \left(\kappa_{xx} + \kappa_{xxx}\right) \pi \\ -2\, \kappa_{zx} \\ 2\, \kappa_{yx} \end{bmatrix}
$$

With (49) – (51) for $\underline{u}_{Dwn}^{B_{Strt}}$, $\underline{u}_{Dwn}^{B_{End}}$, $A_{SFSign}^{B_{Strt}}$, $A_{SFSign}^{B_{End}}$ and (7) for $\lambda_{LinScal}$, λ_{Mis}, λ_{Asym}, $\underline{\lambda}_{Bias}$, (4) becomes for $\delta \hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}}$, $\delta \hat{\underline{a}}_{SF_{End}}^{B_{End}}$:

$$
\delta \hat{\underline{a}}_{SF_{Strt}}^{B_{Strt}} = -g\left(\lambda_{LinScal} + \lambda_{Mis} + \lambda_{Asym} A_{SFSign}^{B_{Strt}}\right) \underline{u}_{Dwn}^{B_{Strt}} + \underline{\lambda}_{Bias}
$$

$$
= -g \begin{bmatrix} \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ \lambda_{yx} & \lambda_{yy} - \lambda_{yyy} & \lambda_{yz} \\ \lambda_{zx} & \lambda_{zy} & \lambda_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \lambda_x \\ \lambda_y \\ \lambda_z \end{bmatrix} = \begin{bmatrix} \lambda_x - g\, \lambda_{xy} \\ \lambda_y - g\left(\lambda_{yy} - \lambda_{yyy}\right) \\ \lambda_z - g\, \lambda_{zy} \end{bmatrix}
$$

(53)

$$
\delta \hat{\underline{a}}_{SF_{End}}^{B_{End}} = -g\left(\lambda_{LinScal} + \lambda_{Mis} + \lambda_{Asym} A_{SFSign}^{B_{End}}\right) \underline{u}_{Dwn}^{B_{End}} + \underline{\lambda}_{Bias}
$$

$$
= -g \begin{bmatrix} \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ \lambda_{yx} & \lambda_{yy} + \lambda_{yyy} & \lambda_{yz} \\ \lambda_{zx} & \lambda_{zy} & \lambda_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} \lambda_x \\ \lambda_y \\ \lambda_z \end{bmatrix} = \begin{bmatrix} \lambda_x + g\, \lambda_{xy} \\ \lambda_y + g\left(\lambda_{yy} + \lambda_{yyy}\right) \\ \lambda_z + g\, \lambda_{zy} \end{bmatrix}
$$

Substituting (52) and (53) with (49) – (50) for $\underline{u}_{Dwn}^{B_{Strt}}$, $C_{B_{End}}^{B_{Strt}}$ in (5) and applying (10) and (11) for a MARS type *B* frame then obtains for $\Delta \hat{\underline{a}}_H^{B_{Strt}}$, $\hat{a}_{Strt_{Down}}^{B_{Strt}}$, $\hat{a}_{End_{Down}}^{B_{End}}$:

$$
\begin{aligned}
&\Delta \hat{\underline{a}}_{H}^{BStrt} = g\, \underline{u}_{Dwn}^{BStrt} \times \underline{\phi}_{End}^{BStrt} + \left(C_{BEnd}^{BStrt}\, \delta \hat{\underline{a}}_{SF\,End}^{BEnd} - \delta \hat{\underline{a}}_{SF\,Strt}^{BStrt} \right)_H \\
&= g \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} \left(\kappa_{xx} + \kappa_{xxx}\right)\pi \\ -2\,\kappa_{zx} \\ 2\,\kappa_{yx} \end{bmatrix} + \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \lambda_x + g\,\lambda_{xy} \\ \lambda_y + g\left(\lambda_{yy} + \lambda_{yyy}\right) \\ \lambda_z + g\,\lambda_{zy} \end{bmatrix} - \begin{bmatrix} \lambda_x - g\,\lambda_{xy} \\ \lambda_y - g\left(\lambda_{yy} - \lambda_{yyy}\right) \\ \lambda_z - g\,\lambda_{zy} \end{bmatrix} \right\}_H \\
&= \begin{bmatrix} 2\,g\,\kappa_{yx} \\ 0 \\ -g\left(\kappa_{xx} + \kappa_{xxx}\right)\pi \end{bmatrix} + \left\{ \begin{bmatrix} \lambda_x + g\,\lambda_{xy} \\ -\lambda_y - g\left(\lambda_{yy} + \lambda_{yyy}\right) \\ -\lambda_z - g\,\lambda_{zy} \end{bmatrix} - \begin{bmatrix} \lambda_x - g\,\lambda_{xy} \\ \lambda_y - g\left(\lambda_{yy} - \lambda_{yyy}\right) \\ \lambda_z - g\,\lambda_{zy} \end{bmatrix} \right\}_H \\
&= \begin{bmatrix} 2\,g\left(\lambda_{xy} + \kappa_{yx}\right) \\ 0 \\ -g\left(\kappa_{xx} + \kappa_{xxx}\right)\pi - 2\,\lambda_z \end{bmatrix} = \begin{bmatrix} 2\,g\left(\mu_{xy} + \upsilon_{xy}/2\right) \\ 0 \\ -2\left[\lambda_z + (\pi g/2)\left(\kappa_{xx} + \kappa_{xxx}\right)\right] \end{bmatrix}
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
\hat{a}_{Strt\,Down}^{BStrt} &= \underline{u}_{Dwn}^{BStrt} \cdot \delta \hat{\underline{a}}_{SF\,Strt}^{BStrt} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda_x - g\,\lambda_{xy} \\ \lambda_y - g\left(\lambda_{yy} - \lambda_{yyy}\right) \\ \lambda_z - g\,\lambda_{zy} \end{bmatrix} = -g\left(\lambda_{yy} - \lambda_{yyy}\right) + \lambda_y \\
\hat{a}_{End\,Down}^{BEnd} &= \underline{u}_{Dwn}^{BEnd} \cdot \delta \hat{\underline{a}}_{SF\,End}^{BEnd} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda_x + g\,\lambda_{xy} \\ \lambda_y + g\left(\lambda_{yy} + \lambda_{yyy}\right) \\ \lambda_z + g\,\lambda_{zy} \end{bmatrix} = -g\left(\lambda_{yy} + \lambda_{yyy}\right) - \lambda_y
\end{aligned}
\tag{55}
$$

The x and z components of $\Delta \hat{\underline{a}}_{H}^{BStrt}$ in (54) and the (55) results for $\hat{a}_{Strt\,Down}^{BStrt}$, $\hat{a}_{End\,Down}^{BEnd}$ are identical to that for sequence 7 in Part 1 [3, Eqs. (16)].

Sequence 7 Unmodelled Gyro Bias Error Effect

With (49) for $\underline{u}_1^{B1,Strt}$ and (6) for $\underline{\kappa}_{Bias}$, (12) for the single +180 deg x axis rotation becomes for $\Delta \underline{\phi}_{GyroBiasRot}^{BStrt}$:

$$\underline{\Delta\phi}_{GyroBiasRot}^{BStrt} = \left[I + \frac{2}{\pi}\left(\underline{u}_1^{B1,Strt}\times\right) + \left(\underline{u}_1^{B1,Strt}\times\right)^2 \right] \frac{\pi}{\dot{\beta}}\,\underline{\kappa}_{Bias}$$

$$= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{2}{\pi}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}^2 \right\} \frac{\pi}{\dot{\beta}} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_z \end{bmatrix} = \begin{bmatrix} \pi\,\kappa_x / \dot{\beta} \\ -2\,\kappa_z / \dot{\beta} \\ 2\,\kappa_y / \dot{\beta} \end{bmatrix} \qquad (56)$$

Substituting (56) in (13) with $\underline{u}_{Dwn}^{BStrt}$ from (49) and C_{BEnd}^{BStrt} from (50) then obtains for $e\left(\Delta\hat{\underline{a}}^{BStrt}\right)_{GyroBias}$:

$$e\left(\Delta\hat{\underline{a}}^{BStrt}\right)_{GyroBias} = g\,\underline{u}_{Dwn}^{BStrt} \times \left\langle \underline{\Delta\phi}_{GyroBias_{Rot}}^{BStrt} + T_{Meas}\left\{ I + F_{Meas}\left(C_{BEnd}^{BStrt} - I\right)\right\}\underline{\kappa}_{Bias} \right\rangle$$

$$= g\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \left\langle \begin{bmatrix} \pi\,\kappa_x / \dot{\beta} \\ -2\,\kappa_z / \dot{\beta} \\ 2\,\kappa_y / \dot{\beta} \end{bmatrix} + T_{Meas}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + F_{Meas}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \right\} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_z \end{bmatrix} \right\rangle \qquad (57)$$

$$= g\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \left\langle \begin{bmatrix} \pi\,\kappa_x / \dot{\beta} \\ -2\,\kappa_z / \dot{\beta} \\ 2\,\kappa_y / \dot{\beta} \end{bmatrix} + \begin{bmatrix} T_{Meas}\,\kappa_x \\ T_{Meas}\left(1 - 2\,F_{Meas}\right)\kappa_y \\ T_{Meas}\left(1 - 2\,F_{Meas}\right)\kappa_z \end{bmatrix} \right\rangle = \begin{bmatrix} 2\,g\left[\kappa_y / \dot{\beta} + T_{Meas}\left(1 - 2\,F_{Meas}\right)\kappa_z / 2\right] \\ 0 \\ -\pi\,g\left(1/\dot{\beta} + T_{Meas}/\pi\right)\kappa_x \end{bmatrix}$$

ROTATION SEQUENCE 13

Rotation sequence 13 is representative of sequences 13 and 14 used to determine the x and y accelerometer bias errors. From Table 1, sequence 13 consists of two +180 degree rotations, the first around IMU axis z from an initial z axis down orientation, the second around IMU axis y:

$$\underline{u}_{Dwn}^{BStrt} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \underline{u}_1^{B1,Strt} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \underline{u}_2^{B2,Strt} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad (58)$$

Then from (3) and (5):

$$C_{B_{1,Strt}}^{B_{Strt}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{+180 about z:} \quad C_{B_{2,Strt}}^{B_{Strt}} = C_{B_{1,Strt}}^{B_{Strt}} C_{B_{2,Strt}}^{B_{1,Strt}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{+180 about y:} \quad C_{B_{End}}^{B_{Strt}} = C_{B_{2,Strt}}^{B_{Strt}} C_{B_{End}}^{B_{2,Strt}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{59}$$

$$\underline{u}_{Dwn}^{B_{End}} = \left(C_{B_{End}}^{B_{Strt}} \right)^T \underline{u}_{Dwn}^{B_{Strt}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

From (58), (59) and the definition of $A_{SFSign}^{B_{Strt}}, A_{SFSign}^{B_{End}}$ as diagonal matrices having elements equal to the negative of $\underline{u}_{Dwn}^{B_{Strt}}$, $\underline{u}_{Dwn}^{B_{End}}$:

$$A_{SFSign}^{B_{Strt}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad A_{SFSign}^{B_{End}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{60}$$

<u>Sequence 13 Measurement</u>

With (58) – (59) and (6) for $\kappa_{LinScal}$, κ_{Asym} , κ_{Mis} , (3) for $\underline{\phi}_{End}^{B_{Strt}}$ becomes for the dual +180 deg rotations:

$$\underline{\phi}_{End}^{B_{Strt}} \approx \sum_i C_{B_{i,Strt}}^{B_{Strt}} \left\{ \left(\kappa_{LinScal} + \kappa_{Asym} \right) \underline{u}_i^{B_{i,Strt}} \pi + 2 \left(\underline{u}_i^{B_{i,Strt}} \times \right) \kappa_{Mis} \underline{u}_i^{B_{i,Strt}} \right\}$$

$$\begin{aligned}
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} \kappa_{xx} + \kappa_{xxx} & 0 & 0 \\ 0 & \kappa_{yy} + \kappa_{yyy} & 0 \\ 0 & 0 & \kappa_{zz} + \kappa_{zzz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \pi + 2 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \kappa_{xy} & \kappa_{xz} \\ \kappa_{yx} & 0 & \kappa_{yz} \\ \kappa_{zx} & \kappa_{zy} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \\
&+ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} \kappa_{xx} + \kappa_{xxx} & 0 & 0 \\ 0 & \kappa_{yy} + \kappa_{yyy} & 0 \\ 0 & 0 & \kappa_{zz} + \kappa_{zzz} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \pi + 2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \kappa_{xy} & \kappa_{xz} \\ \kappa_{yx} & 0 & \kappa_{yz} \\ \kappa_{zx} & \kappa_{zy} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \\
&= \begin{bmatrix} 0 \\ -\left(\kappa_{yy} + \kappa_{yyy} \right) \\ \kappa_{zz} + \kappa_{zzz} \end{bmatrix} \pi + 2 \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \kappa_{xz} \\ \kappa_{yz} \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \kappa_{xy} \\ 0 \\ \kappa_{zy} \end{bmatrix} = \begin{bmatrix} -2 \left(\kappa_{yz} + \kappa_{zy} \right) \\ -\left(\kappa_{yy} + \kappa_{yyy} \right) \pi + 2\, \kappa_{xz} \\ \left(\kappa_{zz} + \kappa_{zzz} \right) \pi - 2\, \kappa_{xy} \end{bmatrix}
\end{aligned} \tag{61}$$

With (58) – (60) for $\underline{u}_{Dwn}^{B_{Strt}}$, $\underline{u}_{Dwn}^{B_{End}}$, $A_{SFSign}^{B_{Strt}}$, $A_{SFSign}^{B_{End}}$ and (7) for $\lambda_{LinScal}$, λ_{Mis}, λ_{Asym}, $\underline{\lambda}_{Bias}$, (4) becomes for $\delta \underline{\hat{a}}_{SF_{Strt}}^{B_{Strt}}$, $\delta \underline{\hat{a}}_{SF_{End}}^{B_{End}}$:

$$\begin{aligned}
\delta \underline{\hat{a}}_{SF_{Strt}}^{B_{Strt}} &= -g \left(\lambda_{LinScal} + \lambda_{Mis} + \lambda_{Asym} A_{SFSign}^{B_{Strt}} \right) \underline{u}_{Dwn}^{B_{Strt}} + \underline{\lambda}_{Bias} \\
&= -g \begin{bmatrix} \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ \lambda_{yx} & \lambda_{yy} & \lambda_{yz} \\ \lambda_{zx} & \lambda_{zy} & \lambda_{zz} - \lambda_{zzz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \lambda_x \\ \lambda_y \\ \lambda_z \end{bmatrix} = \begin{bmatrix} \lambda_x - g\, \lambda_{xz} \\ \lambda_y - g\, \lambda_{yz} \\ \lambda_z - g \left(\lambda_{zz} - \lambda_{zzz} \right) \end{bmatrix} \\
\delta \underline{\hat{a}}_{SF_{End}}^{B_{End}} &= -g \left(\lambda_{LinScal} + \lambda_{Mis} + \lambda_{Asym} A_{SFSign}^{B_{End}} \right) \underline{u}_{Dwn}^{B_{End}} + \underline{\lambda}_{Bias} \\
&= -g \begin{bmatrix} \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ \lambda_{yx} & \lambda_{yy} & \lambda_{yz} \\ \lambda_{zx} & \lambda_{zy} & \lambda_{zz} + \lambda_{zzz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} \lambda_x \\ \lambda_y \\ \lambda_z \end{bmatrix} = \begin{bmatrix} \lambda_x + g\, \lambda_{xz} \\ \lambda_y + g\, \lambda_{yz} \\ \lambda_z + g \left(\lambda_{zz} + \lambda_{zzz} \right) \end{bmatrix}
\end{aligned} \tag{62}$$

Substituting (61) and (62) with (58) – (59) for $\underline{u}_{Dwn}^{B_{Strt}}$, $C_{B_{End}}^{B_{Strt}}$ in (5), and applying (10) and (11) for a MARS type *B* frame, then obtains for $\Delta \underline{\hat{a}}_H^{B_{Strt}}$:

$$\Delta\hat{\underline{a}}_H^{B_{Strt}} = g\,\underline{u}_{Dwn}^{B_{Strt}} \times \underline{\phi}_{End}^{B_{Strt}} + \left(C_{B_{End}}^{B_{Strt}}\, \delta\hat{\underline{a}}_{SF\,End}^{B_{End}} - \delta\hat{\underline{a}}_{SF\,Strt}^{B_{Strt}} \right)_H$$

$$= g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -2\left(\kappa_{yz} + \kappa_{zy}\right) \\ -\left(\kappa_{yy} + \kappa_{yyy}\right)\pi + 2\,\kappa_{xz} \\ \left(\kappa_{zz} + \kappa_{zzz}\right)\pi - 2\,\kappa_{xy} \end{bmatrix} + \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \lambda_x + g\,\lambda_{xz} \\ \lambda_y + g\,\lambda_{yz} \\ \lambda_z + g\left(\lambda_{zz} + \lambda_{zzz}\right) \end{bmatrix} - \begin{bmatrix} \lambda_x - g\,\lambda_{xz} \\ \lambda_y - g\,\lambda_{yz} \\ \lambda_z - g\left(\lambda_{zz} - \lambda_{zzz}\right) \end{bmatrix} \right\}_H$$

$$= g \begin{bmatrix} \left(\kappa_{yy} + \kappa_{yyy}\right)\pi - 2\,\kappa_{xz} \\ -2\left(\kappa_{yz} + \kappa_{zy}\right) \\ 0 \end{bmatrix} + \left\{ \begin{bmatrix} \lambda_x + g\,\lambda_{xz} \\ -\lambda_y - g\,\lambda_{yz} \\ -\lambda_z - g\left(\lambda_{zz} + \lambda_{zzz}\right) \end{bmatrix} - \begin{bmatrix} \lambda_x - g\,\lambda_{xz} \\ \lambda_y - g\,\lambda_{yz} \\ \lambda_z - g\left(\lambda_{zz} - \lambda_{zzz}\right) \end{bmatrix} \right\}_H \tag{63}$$

$$= \begin{bmatrix} g\left(\kappa_{yy} + \kappa_{yyy}\right)\pi - 2\,g\,\kappa_{xz} + 2\,g\,\lambda_{xz} \\ -2\,g\left(\kappa_{yz} + \kappa_{zy}\right) - 2\,\lambda_y \\ 0 \end{bmatrix} = \begin{bmatrix} g\left(\kappa_{yy} + \kappa_{yyy}\right)\pi - g\,\upsilon_{zx} + 2\,g\,\mu_{xz} \\ -2\left(g\,\upsilon_{yz} + \lambda_y\right) \\ 0 \end{bmatrix}$$

The y component of $\Delta\hat{\underline{a}}_H^{B_{Strt}}$ in (63) is identical to that for sequence 13 in Part 1 [3, Eqs. (16)].

Sequence 13 Unmodelled Gyro Bias Error Effect

With (58) for $\underline{u}_1^{B_{1,Strt}}$, $\underline{u}_2^{B_{2,Strt}}$ and (6) for $\underline{\kappa}_{Bias}$, (12) for the dual +180 deg rotations becomes for $\Delta\underline{\phi}_{GyroBiasRot}^{B_{Strt}}$:

$$\Delta\underline{\phi}^{B_{Strt}}_{GyroBiasRot} = \sum_i C^{B_{Strt}}_{B_{i,Strt}} \left[I + \frac{2}{\pi}\left(\underline{u}_i^{B_{i,Strt}}\times\right) + \left(\underline{u}_i^{B_{i,Strt}}\times\right)^2 \right] \frac{\pi}{\dot{\beta}}\, \underline{\kappa}_{Bias}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{2}{\pi} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 \right\} \frac{\pi}{\dot{\beta}} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_z \end{bmatrix}$$

$$+ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{2}{\pi} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}^2 \right\} \frac{\pi}{\dot{\beta}} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_z \end{bmatrix} \qquad (64)$$

$$= \frac{\pi}{\dot{\beta}} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2\,\kappa_y/\pi \\ 2\,\kappa_x/\pi \\ \kappa_z \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2\,\kappa_z/\pi \\ \kappa_y \\ -2\,\kappa_x/\pi \end{bmatrix} \right\} = \begin{bmatrix} -2\left(\kappa_y + \kappa_z\right)/\dot{\beta} \\ \left(2\,\kappa_x - \pi\,\kappa_y\right)/\dot{\beta} \\ \left(\pi\,\kappa_z - 2\,\kappa_x\right)/\dot{\beta} \end{bmatrix}$$

Substituting (64) in (13) with $\underline{u}^{B_{Strt}}_{Dwn}$ from (58) and $C^{B_{Strt}}_{B_{End}}$ from (59), then obtains for $e\left(\Delta\hat{\underline{a}}^{B_{Strt}}\right)_{GyroBias}$:

$$e\left(\Delta\hat{\underline{a}}^{B_{Strt}}\right)_{GyroBias} = g\,\underline{u}^{B_{Strt}}_{Dwn} \times \left\langle \Delta\underline{\phi}^{B_{Strt}}_{GyroBias_{Rot}} + T_{Meas}\left\{ I + F_{Meas}\left(C^{B_{Strt}}_{B_{End}} - I\right)\right\} \underline{\kappa}_{Bias} \right\rangle$$

$$= g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left\langle \begin{bmatrix} -2\left(\kappa_y + \kappa_z\right)/\dot{\beta} \\ \left(2\,\kappa_x - \pi\,\kappa_y\right)/\dot{\beta} \\ \left(\pi\,\kappa_z - 2\,\kappa_x\right)/\dot{\beta} \end{bmatrix} + T_{Meas}\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + F_{Meas}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \right\} \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_z \end{bmatrix} \right\rangle$$

$$= g \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left\langle \begin{bmatrix} -2\left(\kappa_y + \kappa_z\right)/\dot{\beta} \\ \left(2\,\kappa_x - \pi\,\kappa_y\right)/\dot{\beta} \\ \left(\pi\,\kappa_z - 2\,\kappa_x\right)/\dot{\beta} \end{bmatrix} + \begin{bmatrix} T_{Meas}\,\kappa_x \\ T_{Meas}\left(1 - 2\,F_{Meas}\right)\kappa_y \\ T_{Meas}\left(1 - 2\,F_{Meas}\right)\kappa_z \end{bmatrix} \right\rangle \qquad (65)$$

$$= g \begin{bmatrix} -\,g\left[\left(2\,\kappa_x - \pi\,\kappa_y\right)/\dot{\beta} + T_{Meas}\left(1 - 2\,F_{Meas}\right)\kappa_y\right] \\ -2\,g\left[\left(\kappa_y + \kappa_z\right)/\dot{\beta} - T_{Meas}\,\kappa_x/2\right] \\ 0 \end{bmatrix}$$

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