What Do Gyros Measure?

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Gyros are used to measure inertial angular rotation. The term gyro is shorthand for gyroscope, the name originally given to instruments using the gyroscopic inertial properties of a spinning mass as the reference for angular rotation measurements. Today, gyro is the name given generically to any instrument that measures inertial angular rotation [e.g., MEMS (micro-machined electro-mechanical systems) gyros that measure angular rate based on the inertial properties of a vibrating mass, and optical gyros that measure angular rate based on the inertial properties of light]. But what exactly is inertial angular rotation that gyros measure, and what exactly is rotation?

Angular Rotation

Consider two line segments (call them segments A and B) emanating from the same reference point. If the line segments lie on top of one another we call them "parallel". If they are not parallel, we can define the degree of non-parallelism by a parameter called an "angle". For segment B non-parallel to segment A, imagine the angle between A and B being generated from an initial condition in which B is considered to be superimposed on A. While holding the reference point end of segment B stationary, imagine then moving the other end of segment B in a plane from its initial to its actual (final) end-point position. As the end-point of segment B is moved it traces an arc of a circle between A and the final B location. The amount of traced arc is called the "angle" from A to B, measured in radians, and calculated as the linear displacement of the B end-point along the arc divided by the length of segment B. Since the total displacement around a complete circle (i.e., the circumference) equals $2\ \pi$ times the radius of the circle (in this case, the length of B), the angle around a complete circle corresponds to $2\ \pi$ radians.

In a three dimensional world the angle between A and B alone is insufficient to completely define the angular orientation of B relative to A. What is also needed is the orientation of the plane used for the rotation. We can define the plane of rotation by a unit vector $\underline{n}_{AB}$ perpendicular to the rotation plane. The orientation of B relative to A can then be exactly specified as a rotation around the unit vector $\underline{n}_{AB}$ through the rotation angle. "Rotation around the unit vector" is defined by convention to follow the "right hand rule" – pointing the thumb of your right hand in the direction of the unit vector, the rotation is in the direction of your curled right hand fingers in a plane perpendicular to the unit vector.

Notice that with the previous procedure, the orientation of B relative to A can actually be generated by two different rotations, a rotation around $\underline{n}_{AB}$ through a rotation angle (call it rotation angle 1) or a rotation around minus $\underline{n}_{AB}$ (i.e., a unit vector in the opposite direction from $\underline{n}_{AB}$) through another rotation angle that equals 2π minus rotation angle 1. By convention, the rotation angle is specified to be a positive number less than or equal to π

radians with the unit vector for the rotation then selected accordingly to position B correctly after the rotation.

If we define ϕ_{AB} to be the angle between A and B with $\underline{n}_{AB}$ the unit vector for rotation direction, we can mathematically define a "rotation vector" $\underline{\phi}_{AB}$ as the product of ϕ_{AB} (a scalar) with the vector $\underline{n}_{AB}$. With this approach, the orientation of B relative to A is defined to be a rotation around $\underline{\phi}_{AB}$ (using the right hand convention) through an angle equal to the magnitude of $\underline{\phi}_{AB}$.

Angular Rotation Rate

The rotation vector concept can also be used to describe the angular motion of a physical body whose orientation in a dynamic environment can be changing as a function of time. To analyze the body's orientation time history requires an additional concept, the notion of a reference "space" (call it A space) for "viewing" the body's angular orientation at different times. Imagine two images of the body (call it B space) frozen in A space, one image at time t_{m-1} and the other at time t_m. An Euler theorem states (in effect) that there exists one unique rotation vector that will rotate a body from one angular attitude (at time t_{m-1}) to another (at time t_m). The rotation angle equals the rotation vector length (i.e., its magnitude) and is about an imaginary axis fixed in the body and parallel to the rotation vector. Let us denote $\underline{\phi}_{AB}$ as the unique rotation vector for rotating the A space image of body B at time t_{m-1} into the A space image of B at time t_m. (Note that in A space, $\underline{\phi}_{AB}$ describes the angular orientation of space B at time t_m compared to its orientation at time t_{m-1} regardless of the time history of how B actually rotated from its t_{m-1} to t_m angular attitude.)

We can also say that for the time interval Δt from t_{m-1} to t_m, body B has rotated relative to A space at an average "angular rate" of $\underline{\phi}_{AB} / \Delta t$. Letting Δt "go to zero in the limit", we can mathematically define an instantaneous angular rate vector $\underline{\omega}_{AB}$ describing the instantaneous time rate of B Frame rotation relative to A space; i.e., $\underline{\omega}_{AB} \equiv \underline{\phi}_{AB} / \Delta t$ in the limit as $\Delta t \rightarrow 0$.

For a strapdown inertial navigation system installed in a vehicle, three gyros are used to measure the components of the vehicle (B space) angular rate vector $\underline{\omega}_{AB}$. Through an integration process in the system computer, the measured $\underline{\omega}_{AB}$ time history is used to continuously calculate the angular orientation of the vehicle as a function of time. The gyros are mounted at fixed orientation relative to the vehicle ("strapdown") with their input axes mutually orthogonal. Each gyro measures the component of $\underline{\omega}_{AB}$ along its input axis (i.e., the mathematical dot product between $\underline{\omega}_{AB}$ and an imaginary unit vector along the gyro input axis). The gyros measure vehicle (B space) angular rate relative to reference space A, but what is reference space A? For a gyro, there is only one A space; a gyro measures angular rotation relative to non-rotating inertial space. What does this mean? How do we define non-rotating space without referring to another reference space, and how does a gyro know which space is not rotating?

Local Non-Rotating Inertial Space

Consider a virtual (imaginary) space (call it space A) defined by several independent virtual point masses in a uniform gravity field. Assume that there are no forces to act on the masses (i.e., a "force-free" zone). If the relative velocity between the masses is zero, all masses will then experience the same velocity change due to gravity, hence, their relative positions will remain constant. We could draw lines from mass 1 to mass 2 and from mass 1 to mass 3 and measure the angle between the lines. Because the relative position between all the masses is constant, the angle so measured will also be constant as would angles between any position lines formed in the same way between these and other point masses.

Let us now consider a duplicate force-free space (call it space B) that overlaps space A, has the same uniform gravity field as space A, and is defined by a duplicate set of real masses, initially in the same position location and velocity as the space A virtual masses. The position of the space B masses relative to one another will then continue to remain constant (as did the space A masses). In addition, the relative position of the space B real masses will also remain constant relative to the space A virtual masses. If we draw lines between space B masses and measure their angular orientation relative to lines between space A masses, they will also be constant, hence, there will be no angular rate between space B and space A.

Now, let us suddenly connect the real space B masses together using mass-less rods to form a rigid body. Because there was no relative velocity between the masses before they were connected, there will be no forces transmitted along the rods after they are connected, and the masses will behave as a group as if the connecting rods didn't exist. Thus, the angular orientation between the space B rigid body masses and lines drawn between space A masses will continue to remain constant (i.e., no angular rate between rigid body B and space A).

What if we now apply a force on rigid body B. Then, from Newtonian dynamics we know that in addition to its gravitational acceleration, rigid body B will develop a linear acceleration (rate of change of linear velocity) in proportion to the applied force. It will also develop an angular acceleration (rate of change of its angular rate) proportional to the resulting applied torque (the product of the applied force and the perpendicular distance of the line of force to the body B center of mass). The angular acceleration will integrate into angular rate that will change the angular orientation of B from its attitude before the force was applied. The rigid body motion so described is the combined effect of accelerations of all the masses in the rigid body in response to forces transmitted along the connecting rods that maintain fixed positions between masses in the presence of applied force on a particular portion of the body. The net result is that the forces create body B rotation relative to reference space A. Thus, virtual force free space A acts as a local angular "inertial reference space" for measuring body B angular rotation created by applied force. By definition we denote this as a local non-rotating inertial space.

General Non-Rotating Inertial Space

Does the space A local angular inertial reference concept extend to other regions of space with a different uniform gravity field? To compare relative inertial angular properties between remotely located spaces that might be moving relative to one another we can use the concept that general relative motion consists of two independent elements, relative translation and relative rotation of one space relative to the other. By relative translation between spaces, we mean that every point in one space (that by definition is stationary relative to other points in the same space) is at the same relative velocity relative to every point in the other space. Relative rotation between spaces can be measured by the degree to which line segments fixed in one space remain parallel to line segments fixed in the other space that are initially drawn parallel. If over time, at least two non-parallel line segments in one space remain parallel to the corresponding set of line segments in the other space, we can say that there is no angular rotation rate between the two spaces. By extension then, either space can be used to represent the reference for measuring angular rotation of any third space.

Distant from space A, consider another space A* in the same uniform gravity field as space A and defined by a separate set of A* point masses at zero initial velocity relative to one another. As in space A, the space A* masses will maintain their relative positions over time because they all have the same gravitational acceleration in A* space. In general, at some arbitrary initial time, there may be relative velocity between spaces A and A*. Because all space A masses are at zero relative velocity (and similarly for the space A* masses) all space A* mass points will have the same velocity relative to any space A mass point.

In space A, now draw line segments from mass points A1 to A2 (call it segment A12) and from A3 to A4 (call it segment A34). Select arbitrary mass points A*1 and A*3 in space A* and from them, draw lines that are parallel to segments A12 and A34. Add an A* fixed mass point A*2 onto the line from A*1 to form line segment A*12 between A*1 and A*2 that is parallel to segment A12. Similarly, add A*4 onto the line from A*3 to form line segment A*34 that is parallel to segment A34. Because all points in space A* are moving at the same velocity relative to space A points, the two A* line segments will remain parallel to their counterpart space A line segments as the mass points on the segments move at the same velocity relative to one another. Thus there will be zero relative angular rate between spaces A and A*, and the relative motion between the spaces is classified as pure translation.

Now consider the same A, A* situation, but further generalize to allow that A and A* are in different uniform gravity fields. As described earlier, each mass point in space A will remain at fixed position relative to other mass points in space A (and similarly for the relative position of space A* mass points), but the relative velocity between space A and A* will now be changing at a rate equal to the difference between the A and A* gravity fields. Then every point in space A* will be changing its velocity by the same amount relative to the space A mass points, and the A12, A34 and A*12, A*34 line segments will continue to remain parallel. Thus, for the more general case where spaces A and A* are in different uniform gravity fields and initially at different velocities, the rotation rate between A and A* will still be zero.

We can conclude that spaces A and A* are equivalent; either can be used as a "non-rotating inertial space" for referencing the angular rotation rate of any third space created by applied forces. As such, general non-rotating inertial space can be defined for force generated angular rotation of a physical body, as any space in the universe defined by three or more virtual masses in a uniform gravity field, having zero relative velocity between the masses at one point in time, and having zero force applied to any of the masses.

Angular Rate Sensing By Mechanical Gyros

Forces applied to a rigid body generate linear and angular acceleration proportional to the applied forces. While the forces are being applied, internal forces are developed between the body masses proportional to the applied forces, hence proportional to the resulting body linear and angular acceleration. Based on measurements of the internal forces, mechanical gyros mounted within a body measure body angular rate, not angular acceleration. How do they do this? The answer is that within each gyro are proof masses that are driven into linear motion (i.e., velocity) relative to the gyro case by a motor within the gyro. Gyro case angular rate produces reaction forces on the moving proof masses that then form a measurable composite for generating the gyro angular rate output signal.

The motor drive in a gyro is arranged so that for each generated proof mass velocity in one direction (relative to the gyro case), an equal but oppositely directed relative velocity is generated on another proof mass. Force applied to the proof mass pairs will produce a change in their velocities in the direction of the applied force (i.e., an acceleration) relative to non-rotating inertial space. Conversely, for acceleration of the proof masses relative to inertial space, forces must be applied to the proof masses to generate the acceleration. The Appendix derives the relationship between relative proof mass acceleration and proof mass motion relative to the gyro case, a space that may be rotating relative to "non-rotating inertial space". Relative to the gyro case, the appendix shows that for equal mass magnitudes in a proof mass pair, the difference in forces applied to the proof masses is proportional to:

$$a_{SF_{12u}} = \left(\underline{\omega}_{IG} \cdot \underline{u}_G\right)\left(\underline{\omega}_{IG} \cdot \underline{r}_{12}\right) + \dot{\underline{\omega}}_{IG} \cdot \left(\underline{r}_{12} \times \underline{u}_G\right) + 2\,\underline{\omega}_{IG} \cdot \left(\underline{v}_{12} \times \underline{u}_G\right) \qquad (1)$$

where $\underline{\omega}_{IG}$ is the gyro case angular rate relative to inertial space, $\dot{\underline{\omega}}_{IG}$ is the time rate of change of $\underline{\omega}_{IG}$ relative to the gyro case, $\underline{r}_{12}$ is the position vector from mass 1 to mass 2 (in the proof mass pair), $\underline{v}_{12}$ is the velocity of proof mass 2 relative to proof mass 1, $\underline{u}_G$ is a unit vector perpendicular to the plane containing $\underline{r}_{12}$ and $\underline{v}_{12}$ (or perpendicular to the $\underline{r}_{12}$, $\underline{v}_{12}$ vectors if they are collinear), and $a_{SF_{12u}}$ is the component along $\underline{u}_G$ of the normalized difference between forces applied to masses 1 and 2 (normalized by their mass magnitudes and also known as specific force). The $2\,\underline{\omega}_{IG} \cdot \left(\underline{v}_{12} \times \underline{u}_G\right)$ term in Equation (1) is proportional to the component of gyro case angular rate around axis $\underline{u}_G$, which is the term the gyro is to measure. The measurement is the composite effect of $a_{SF_{12u}}$ forces on all moving proof masses within the gyro case as registered on a composite force output transducer. The remaining terms in Equation (1) are error terms that are minimized or eliminated by the gyro construction (e.g., eliminated by mass symmetry so that similar terms for different mass point pairs are of the same magnitude but opposite in sign). The primary method for minimizing the remaining error terms is to make $\underline{v}_{12}$ large.

For a spinning mass gyro, the proof masses comprise a spinning rotor and the proof mass velocities are the velocities of points on the rotor relative to the spin axis. The gyro output is measured torque (the composite of dual oppositely directed proof mass forces) applied to the rotor to maintain the spin axis aligned to the case in the presence of rotation. The proof mass pairs are points on opposite sides of the rotor equidistant from the axis. Because of rotor symmetry around the spin axis, the orientation of proof mass pairs and their relative velocity appears constant relative to the gyro case (relative to the gyro case, previous proof mass pairs are constantly being replaced by identical pairs at the same velocity). The result is that the composite torque being measured provides a continuous measurement of gyro case angular rate.

For a MEMS type gyro, the proof masses are linearly vibrating elements which makes the $\underline{v}_{12}$ term oscillatory. For this type of situation, the composite force measurement is also oscillatory at the vibrating mass frequency, with amplitude proportional to the gyro case angular rate. The angular rate measurement is then obtained by demodulating the oscillatory signal.

Angular Rate Sensing By Optical Gyros

Optical gyros (ring laser gyros - RLGs, and fiber optic gyros - FOGs) measure angular rate using the inertial properties of light. Both RLGs and FOGs are based on the Sagnac effect (derivable from General Relativity) for oppositely directed light beams traversing the same closed path. Under angular rotation about an axis perpendicular to a closed light path, relative to non-rotating inertial space, the apparent closed path length for light traversing the closed path in the direction of rotation will be longer than for light traversing the same closed path in the opposite direction. If the oppositely directed light beams are of the same frequency, the result will be a net change in phase shift between the two beams for each traversal of the of the closed path. The phase shift is proportional to the angular rate, the signal to be measured by the optical gyro. Moreover, the angular rate measured is relative to non-rotating inertial space, presumably of the same type as described previously (an independent analysis by an optical gyro theoretician would be beneficial to confirm this assumption).

For a FOG, the light beam path is created with a circularly-wound fiber optic coil and the light source is a super-luminescent diode. Light from the diode enters a linear segment of fiber optic material which is spliced onto the fiber optic coil. The splice causes the light to split into "clockwise" and "counter-clockwise" beams which then traverse the coil length in opposite directions. A second splice at the end (start) of the coil recombines the beams onto an output linear fiber optic segment that terminates at a photodiode, the device output transducer. The phase shift (proportional to the angular rate) in the recombined beams produces a change in light intensity, the signal measured by the photo diode. The sensitivity of the FOG (i.e., phase shift per unit of angular rate) is proportional to the length of the fiber coil (e.g., 400 meters in typical FOGs).

For an RLG, the closed light beam path is created by reflecting mirrors (3 mirrors for a triangular closed path and 4 for a square closed path). The mirrors are mounted to a Zerodur structure (a thermally stable translucent material) containing tubular space between mirrors (the "cavity") for the light to pass. The resulting mirror-closed cavity also serves to house a helium-neon gas ring laser, the source of the single frequency light beams that traverse the cavity in "clockwise" and "counter-clockwise" directions. As with the FOG, for each closed-loop traversal of the light beams, a phase shift develops in the RLG between the oppositely directed light beams. Unlike the FOG, because of its basic construction as a laser, the beams continue to re-traverse the same path continuously, thereby adding phase shift for each traversal. Thus for the RLG, the phase shift between the two light beams develops a rate of change proportional to the input angular rate, and the phase shift becomes proportional to the integrated angular rate.

Phase shift output from the RLG is obtained by allowing a small fraction of each beam to escape the cavity and be recombined on a photodiode. Under input angular rate, the photodiode outputs a sinusoidal signal with each wave representing a phase change of 2π radians. Each photodiode output cycle represents a corresponding increment of integrated gyro case input angular rate (e.g., 2 arc sec angular motion per photodiode output cycle).

The Effect of Gravity Gradient On Gyro Output

Mechanical and optical gyros measure angular rate relative to non-rotating inertial space. General non-rotating inertial space as previously defined included a general uniform gravity field within the space. Similarly, our analysis of what gyros measure was also based on the gyro being in a uniform gravity field. But what if the gravity field surrounding the gyro is not uniform. Will this affect the gyro output? The answer depends on the shape of the gravity field and the type of gyro being used.

Gravity Field Shape

For most applications, the assumption of gravity field uniformity is a very good approximation, particularly for applications where the gyro is used at a great distance from planets. When near planets, a better approximation is that gravity can be approximated as a spherical field pointing toward the center of the planet with magnitude inversely proportional to the square of the distance from the planet's center (i.e., Newton's gravitational law recognizing that for a spherical body of uniform density, gravity above the body's surface is exactly what it would be if the mass of the body was concentrated at the center of the body).

Spinning-Mass Gyros In A Spherical Gravity Field

For a spinning mass gyro, Newtonion physics shows that torque T applied to the spinning rotor around an axis perpendicular to the spin axis will produce a precessional rate of the rotor given by

$$T = \Omega_{Precess} \, H_{Rotor} \tag{2}$$

in which

$$H_{Rotor} = I_{Rotor}\,\omega_{Rotor} \tag{3}$$

where $\Omega_{Precess}$ is the precessional rate created by the applied torque, H_{Rotor} is the rotor angular momentum, I_{Rotor} is the moment of inertia of the rotor about its the spin axis, and ω_{Rotor} is the rotor spin rate. Equation (2) is the equivalent of the Equation (1) $2\,\underline{\omega}_{IG} \cdot \left(\underline{u}_G \times \underline{v}_{12}\right)$ term for all combined mass points on the rotor (and neglecting the remaining terms as small or compensatable). For a strapdown gyro for which T maintains rotor alignment with the gyro case, $\Omega_{Precess}$ is the component of gyro case angular rate $\underline{\omega}_{IG}$ along the gyro input axis. Hence, T is proportional to gyro input axis angular rate.

Equation (2) applies for a uniform gravity field for which $\Omega_{Precess}$ is angular rate relative to general non-rotating space as defined earlier (i.e., a non-rotating space in a general uniform gravity field that may differ from the field surrounding the gyro). In actuality, uniform gravity is an approximation. For a spherical gravity field (like the gravity field surrounding the earth) the gravity gradient effect across an axially mass symmetric body (like a gyro rotor) will generate a net torque T_{grav} on the body according to the formula[1]:

$$T_{grav} = \frac{3}{2}\frac{g}{R}\left(I_{Axial} - I_{Cross}\right)\sin 2\theta \tag{4}$$

where g is gravity magnitude at the body's center of mass, R is the distance of the center of mass from the center of the earth, I_{Axial} is the moment of inertia of the body about its center of mass around the axis of symmetry, I_{Cross} is the body moment of inertia about its center of mass around an axis perpendicular to the symmetric axis, θ is the angle between vertical and the axis of symmetry, and T_{grav} is perpendicular to the plane defined by the axis of symmetry and vertical. For a solid cylindrical body, the moments of inertia are given by[2]

$$I_{Axial} = \frac{1}{2}\,m\,r^2 \qquad I_{Cross} = \frac{1}{12}\,m\left(3\,r^2 + l^2\right) \tag{5}$$

where m is the body mass, r is the cylinder radius and l is the cylinder length.

Identifying the cylindrical body in (4) and (5) as the spinning wheel gyro rotor (i.e., $I_{Axial} = I_{Rotor}$), $\Delta\Omega_{grav}$ as the portion of $\Omega_{Precess}$ in Equation (2) that would be generated in response to the T_{grav} component in T, Equations (2) – (5) combined then finds for $\sin 2\theta$ equal to a maximum of 1:

$$\Delta\Omega_{grav} = \frac{3}{4}\left(1 - \frac{1}{3}\frac{l^2}{r^2}\right)\frac{\omega_S}{\omega_{Rotor}}\,\omega_S \qquad \omega_S \equiv \sqrt{\frac{g}{R}} \tag{6}$$

To maintain the gyro rotor aligned to the case, an additional torque T_{grav} (an error torque) will be applied within the gyro in opposition to the Equation (4) gravity gradient torque.

The gyro output signal proportional to the torque (i.e., the inverse of equation (2)) will thereby contain an error equal to the negative of $\Delta\Omega_{grav}$ in Equation (6).

Inertial navigation analysts will recognize ω_S in Equations (6) to be the Schuler frequency which on the surface of the earth equals one revolution in 84 minutes or 257 deg/hr. A typical rotor spin rate for a high performance spinning mass gyro is 24,000 revolutions per minute[3]. Based on these figures, the gravity gradient induced error from Equations (6) for l = r is 0.000064 deg/hr. For an inertial navigation system (INS) with 1 nautical mile per hour (nmph) position error build-up rate (typical for an aircraft INS), the average gyro error contributing to the 1 nmph position drift is 0.01 deg/hr. As such, the 0.000064 gravity gradient effect is negligible. For high precision applications, the gyro output can be compensated (corrected) for gravity gradient induced error

MEMS Gyros In A Spherical Gravity Field

For a MEMS type gyro, gravity gradient produces a differential specific force between the counter-vibrating linear elements. Because the vibration distance is small compared to the distance between vibrating elements, the differential force is approximately constant. The basic angular rate measurement signal in a MEMS gyro is modulated at the moving element linear vibration frequency. The gyro output is obtained by demodulating this signal. The demodulation process also removes any constant differential specific force component caused by gravity gradient. This discussion is somewhat academic because any residual gravity effect in a MEMS type gyro output would be negligible compared to MEMS gyro accuracy capabilities.

Optical Gyros In A Spherical Gravity Field

To the author's knowledge, the effect of gravity gradient across RLGs or FOGs has not been analyzed. Presumably it would require the application of General Relativity theory for a definitive answer. Based on the preceding analysis for spinning wheel gyros, it might be surmised that the effect would be negligible for most applications. However, for high performance applications where the physical size of the gyros can be quite large (e.g., FOGs in shipboard INSs), an analysis of the effect is warranted.

Universal Non-Rotating Inertial Space

The general model given earlier for non-rotating inertial space was based on the behavior of a group of independent virtual mass points with zero relative velocity in a uniform gravity field. In the section defining angular rotation rate, it was shown that in the absence of applied torques, the mass points could be connected by rigid mass-less rods with the resulting rigid body then maintaining the same inertially non-rotating property as the free mass points. Based on this concept and Equation (4) we can now further generalize the definition of non-rotating inertial space to encompass both and spherical gravity fields. The method is to define the space as being force-free and occupied by a virtual rigid body of symmetric mass such that $I_{Cross} = I_{Axial}$. With such an arrangement, Equation (4) shows that the gravity gradient torque on the virtual body will be zero. If the angular rate for this body relative to general non-rotating space (as defined earlier in a uniform gravity field) is initialized to be zero, the angular rate will thereby remain at zero. Thus, a virtual mass symmetric rigid body can be used as a non-rotating inertial reference in uniform and in spherical gravity fields. Since all gravity fields near-to or distant from planets or stars can be categorized to a high degree of accuracy as uniform or spherical (as discussed earlier), this concept is applicable to all universal applications.

Appendix

Relative Motion Between Two Moving Masses On A Translating/Rotating Base

Consider a point at position vector $\underline{R}_0$ relative to a stationary reference point in non-rotating inertial coordinate frame I. Also consider a point mass at position $\underline{r}$ relative to $\underline{R}_0$ such that

$$\underline{R} = \underline{R}_0 + \underline{r} \tag{A-1}$$

Projecting Equation (A-1) on frame I axes finds for the components:

$$\underline{R}^I = \underline{R}_0^I + \underline{r}^I \tag{A-2}$$

where $[\,]^{(\,)}$ is the column matrix formed from the components of vector [ ] projected on coordinate frame ( ) axes. We also define another coordinate frame G in which the end point of vector $\underline{R}_0$ is stationary, and whose angular orientation relative to frame I can be represented by the direction cosine matrix C_G^I. Describing the components of $\underline{r}$ in frame G, Equation (A-2) becomes

$$\underline{R}^I = \underline{R}_0^I + C_G^I\, \underline{r}^G \tag{A-3}$$

Let coordinate frame I, the $\underline{R}_0$ point, and the point mass position $\underline{R}$ be in the same uniform gravitational field. Then the second derivative of $\underline{R}^I$ is the specific force acceleration imposed on the point mass:

$$\ddot{\underline{R}}^I = \underline{a}_{SF}^I \tag{A-4}$$

An equivalent to (A-4) is derived by taking two successive derivatives of (A-3). The first derivative finds

$$\dot{\underline{R}}^I = \dot{\underline{R}}_0^I + \dot{C}_G^I\, \underline{r}^G + C_G^I\, \dot{\underline{r}}^G \tag{A-5}$$

Defining the angular rate of frame G relative to frame I as $\underline{\omega}_{IG}$, it is well known that

$$\dot{C}_G^I = C_G^I \left(\underline{\omega}_{IG}^G \times\right) \tag{A-6}$$

where $\left\{[\ ]^{(\ )}\times\right\}$ is the matrix cross-product operator form of the column matrix $[\ ]^{(\ )}$ defined implicitly as the matrix that when multiplied with the column matrix form of an arbitrary vector $\underline{V}$ satisfies $\left\{[\ ]^{(\ )}\times\right\}\underline{V}^{(\ )} = [\ ]^{(\ )}\times\underline{V}^{(\ )} = \left\{[\ ]\times\underline{V}\right\}^{(\ )}$. We also identify $\dot{\underline{r}}^G$ in (A-5) as the G frame components of point mass velocity $\underline{v}$ relative to the $\underline{R}_0$ end point (previously defined to be stationary in frame G)

$$\dot{\underline{r}}^G = \underline{v}^G \tag{A-7}$$

With (A-6) and (A-7), (A-5) becomes

$$\dot{\underline{R}}^I = \dot{\underline{R}}_0^I + C_G^I \left(\underline{\omega}_{IG}^G \times\right) \underline{r}^G + C_G^I \underline{v}^G \tag{A-8}$$

Taking the derivative of (A-8) and substituting (A-6), (A-7) and their derivatives where appropriate then obtains

$$\begin{aligned}\dot{\underline{R}}^I = \ddot{\underline{R}}_0^I &+ C_G^I \left[\underline{\omega}_{IG}^G \times \left(\underline{\omega}_{IG}^G \times \underline{r}^G\right)\right] + C_G^I \left(\dot{\underline{\omega}}_{IG}^G \times \underline{r}^G\right) \\ &+ 2\, C_G^I \left(\underline{\omega}_{IG}^G \times \underline{v}^G\right) + C_G^I \dot{\underline{v}}^G\end{aligned} \tag{A-9}$$

Finally, we substitute (A-4) for $\ddot{\underline{R}}^I$ and multiply the result by C_I^G:

$$\underline{a}_{SF}^G = C_I^G \ddot{\underline{R}}_0^I + \underline{\omega}_{IG}^G \times \left(\underline{\omega}_{IG}^G \times \underline{r}^G\right) + \dot{\underline{\omega}}_{IG}^G \times \underline{r}^G + 2\left(\underline{\omega}_{IG}^G \times \underline{v}^G\right) + \dot{\underline{v}}^G \tag{A-10}$$

Now consider two proof masses in frame G. Each satisfies Equation (A-10):

$$\begin{aligned}\underline{a}_{SF_1}^G &= C_I^G \ddot{\underline{R}}_0^I + \underline{\omega}_{IG}^G \times \left(\underline{\omega}_{IG}^G \times \underline{r}_1^G\right) + \dot{\underline{\omega}}_{IG}^G \times \underline{r}_1^G + 2\left(\underline{\omega}_{IG}^G \times \underline{v}_1^G\right) + \dot{\underline{v}}_1^G \\ \underline{a}_{SF_2}^G &= C_I^G \ddot{\underline{R}}_0^I + \underline{\omega}_{IG}^G \times \left(\underline{\omega}_{IG}^G \times \underline{r}_2^G\right) + \dot{\underline{\omega}}_{IG}^G \times \underline{r}_2^G + 2\left(\underline{\omega}_{IG}^G \times \underline{v}_2^G\right) + \dot{\underline{v}}_2^G\end{aligned} \tag{A-11}$$

Then the difference between the specific forces at the two mass points is

$$\underline{a}_{SF_{12}}^G = \underline{\omega}_{IG}^G \times \left(\underline{\omega}_{IG}^G \times \underline{r}_{12}^G\right) + \dot{\underline{\omega}}_{IG}^G \times \underline{r}_{12} + 2\,\underline{\omega}_{IG}^G \times \underline{v}_{12} + \dot{\underline{v}}_2^G - \dot{\underline{v}}_1^G \tag{A-12}$$

where $\underline{a}_{SF_{12}} \equiv \underline{a}_{SF_2} - \underline{a}_{SF_1}$, $\underline{v}_{12} \equiv \underline{v}_2 - \underline{v}_1$, and $\underline{r}_{12} \equiv \underline{r}_2 - \underline{r}_1$.

For mechanical gyros (spinning mass or MEMS), the G frame is fixed relative to the gyro case, and the $\underline{r}_1^G, \underline{r}_2^G, \underline{v}_1^G, \underline{v}_2^G, \dot{\underline{v}}_2^G, \dot{\underline{v}}_1^G$ vectors describe the position and motion of mass points on a moving element within the gyro that are either in the same plane or collinear. Let’s define a unit vector perpendicular to this plane (or line) as $\underline{u}_G$. Then $a_{SF_{12u}}$, the component of $\underline{a}_{SF_{12}}$ along $\underline{u}_G$ is

$$a_{SF_{12u}} = \underline{u}_G^G \cdot \left[\underline{\omega}_{IG}^G \times \left(\underline{\omega}_{IG}^G \times \underline{r}_{12}^G\right) + \dot{\underline{\omega}}_{IG}^G \times \underline{r}_{12} + 2\,\underline{\omega}_{IG}^G \times \underline{v}_{12}\right] \qquad \text{(A-13)}$$

After application of suitable vector product identities and recognizing that the dot product between two vectors is identical for any coordinate frame in which it is evaluated, we find for (A-13):

$$a_{SF_{12u}} = \left(\underline{\omega}_{IG} \cdot \underline{u}_G\right)\left(\underline{\omega}_{IG} \cdot \underline{r}_{12}\right) + \dot{\underline{\omega}}_{IG} \cdot \left(\underline{r}_{12} \times \underline{u}_G\right) + 2\,\underline{\omega}_{IG} \cdot \left(\underline{v}_{12} \times \underline{u}_G\right) \qquad \text{(A-14)}$$

Equation (A-14) defines the component along $\underline{u}_G$ of the relative specific force between two mass points on a moving element within the gyro. The composite effect of (A-14) for all mass point pairs on the gyro's moving element is the output measurement generated by the gyro output transducer.

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