MODERN STRAPDOWN ATTITUDE ALGORITHMS AND THEIR ACCURACY, VERSUS ACCURACY REQUIREMENTS FOR UNAIDED STRAPDOWN INERTIAL NAVIGATION

Paul G. Savage
Strapdown Associates, Inc.
Maple Plain, MN 55359 USA

WBN-14025
www.strapdowassociates.com
February 9, 2020

ABSTRACT

The analytical structure of modern-day strapdown attitude updating algorithms is described for application to current and advanced unaided strapdown inertial navigation for aircraft. Algorithm accuracy requirements are presented for compatibility with current and future projected inertial navigation system (INS) needs, limited in part, by fundamental uncertainty limits in INS gravity modeling and physical sensor-to-sensor alignment calibration. The performance capabilities of current-day strapdown attitude algorithms are described, demonstrating accuracy exceeding forecasted future INS requirements.

INTRODUCTION

Strapdown attitude algorithm design originally focused on circumventing limitations in early airborne computer technology (in throughput, memory, and fixed-point word-length). Thus, the invention of the two-speed approach [2] in which a basic attitude updating operation is performed using a higher-order algorithm, with input from a high-speed simpler lower-order algorithm designed to measure the effects of high frequency angular vibration. The two-speed approach was first used in 1975 - 1985 for strapdown inertial navigation aircraft application, employing a fifth-order "exact" direction cosine attitude updating operation [4], coupled with a high-speed two-sample Taylor time-series based algorithm implemented by special-purpose digital electronics within the computer I/O (i.e., the original implementation of $\delta\phi_{Algo_m}$ in (5) to follow). Since then, computer technology has evolved to a floating-point architecture whereby throughput/memory/word-length limitations are no longer an issue, and all computational functions reside within the main processor. The two-speed approach continues to be used today for software design flexibility, allowing attitude update rate selection based on output interface convenience and accuracy under maneuvering, with the high speed computation portion selected for accuracy under vibration.

No longer being constrained by computer limitations, algorithm design has blossomed into a competitive technical field for improved accuracy compared to past algorithms under hypothesized angular motion, coning being a common vehicle for performance evaluation. Given the fundamental accuracy limitation of strapdown inertial sensors, gravity uncertainty, and achievable sensor-to-sensor physical alignment, the new algorithm accuracies are generally much higher than needed in practice.

This article describes the analytical structure of modern-day attitude algorithms used in a strapdown INS for aircraft. Algorithm accuracy requirements are derived for compatibility with the forecasted needs in advanced future INSs. The accuracy of existing strapdown attitude updating algorithms is then presented, demonstrating their ability to exceed the forecasted advanced INS requirements.

THE STRUCTURE OF MODERN-DAY STRAPDOWN ATTITUDE ALGORITHMS

The modern-day strapdown INS attitude updating approach is based on an exact two-speed structure of the typical form:

$$\boldsymbol{C}_n = \boldsymbol{C}_{n-1}\,\boldsymbol{B}_n$$

$$\boldsymbol{B}_n = \boldsymbol{I} + \frac{\sin\sigma_n}{\sigma_n}(\boldsymbol{\sigma}_n\times) + \frac{1-\cos\sigma_n}{\sigma_n^2}(\boldsymbol{\sigma}_n\times)(\boldsymbol{\sigma}_n\times) \qquad \sigma_n \equiv \sqrt{\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_n} \tag{1}$$

$$\frac{\sin\sigma_n}{\sigma_n} = 1 - \frac{1}{3!}\sigma_n^2 + \frac{1}{5!}\sigma_n^4 + \cdots \qquad \frac{1-\cos\sigma_n}{\sigma_n^2} = 1 - \frac{1}{2!}\sigma_n^2 + \frac{1}{4!}\sigma_n^4 + \cdots$$

where n is the attitude update cycle rate index (e.g., 200 Hz), $\boldsymbol{C}_n$ is an attitude direction cosine matrix at the end of the n cycle, $\boldsymbol{B}_n$ is a direction cosine matrix representing the change in attitude over an n cycle, $\boldsymbol{I}$ is the identity matrix, $\boldsymbol{\sigma}_n$ is a rotation vector equivalent of $\boldsymbol{C}_n$, and $(\boldsymbol{\sigma}_n\times)$ is the cross-product matrix equivalent of $\boldsymbol{\sigma}_n$ that for arbitrary vector $\boldsymbol{V}$ satisfies $(\boldsymbol{\sigma}_n\times)\boldsymbol{V} = \boldsymbol{\sigma}_n\times\boldsymbol{V}$. Continued advances in computer throughput, long-word-length floating-point arithmetic, and memory capacity allow the series expansions in (1) to be carried out to high order (e.g., 15 terms), making (1) virtually error free (save for algorithmic errors in the calculation of $\boldsymbol{\sigma}_n$). Rotation vector $\boldsymbol{\sigma}_n$ in (1) is calculated based on the commonly used approximation:

$$\boldsymbol{\sigma}_n \approx \Delta\boldsymbol{\theta}_n + \delta\boldsymbol{\sigma}_n \qquad \Delta\boldsymbol{\theta}(t) \equiv \int_{t_{n-1}}^{t}\boldsymbol{\omega}\,d\tau \qquad \Delta\boldsymbol{\theta}_n = \Delta\boldsymbol{\theta}(t = t_n) \qquad \delta\boldsymbol{\sigma}_n \approx \frac{1}{2}\int_{t_{n-1}}^{t_n}\Delta\boldsymbol{\theta}(t)\times\boldsymbol{\omega}\,dt \tag{2}$$

where $\boldsymbol{\omega}$ is angular rate (provided from gyro measurements), $\Delta\boldsymbol{\theta}(t)$ is integrated angular rate from the start of an n cycle to a general time t within the n cycle, $\Delta\boldsymbol{\theta}_n$ is integrated angular rate over the n cycle, and $\delta\boldsymbol{\sigma}_n$ is a "coning correction" used to correct $\Delta\boldsymbol{\theta}_n$ for its variation from $\boldsymbol{\sigma}_n$. The (2) integral formula approximates an integral of the exact rotation vector rate equation:

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\sigma}\times\boldsymbol{\omega} + \frac{1}{\sigma^2}\left[1 - \frac{\sigma\sin\sigma}{2(1-\cos\sigma)}\right]\boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\boldsymbol{\omega}) \tag{3}$$

where σ is the magnitude of $\boldsymbol{\sigma}$. Note: Rotation vector rate equation (3) was first derived by Laning in 1949 [1], and is specifically credited as such by Bortz as his first reference in [3].

The two-speed architecture represented by (1), is built on the well-known exact solution for attitude updating when the $\Delta\boldsymbol{\theta}_n$ rotation axis direction is constant, plus a small correction $\delta\boldsymbol{\sigma}_n$ as promoted by Jordan [4]. The new architecture improved the accuracy of the original two-speed approach in [2] which was based on a truncated second-order Picard expansion for $\boldsymbol{B}_n$. The exact (1) structure allows for any form of attitude that has a closed-form non-singular exact equivalency to the $\boldsymbol{\sigma}_n$ rotation vector. In addition to the commonly used direction cosine approach depicted in (1), an equivalent version based on the more politically correct attitude quaternion is also commonly used [Savage, P.G., "Geordie's Quaternion Decision", SAI-WBN-14014, Feb 17, 2016 http://www.strapdownassociates.com/Geordie's%20Quaternion.pdf]. Other variations are also possible.

It is to be noted that because direction cosines or any other attitude representation in (1) will be exact, errors commonly attributed to attitude updating algorithms (e.g., normalization/ orthogonalization errors in direction cosines or normalization errors in a quaternion) will be zero, hence, need not be of consideration in attitude form selection for (1). (Note: A direction cosine (or quaternion) updating operation in (1) will retain exact normalization/orthogonalization accuracy, regardless of any error that may exist in the computation of input $\boldsymbol{\sigma}_n$.) Additionally, having the dominant portion of (1) exact, algorithm software validation is greatly facilitated (by comparison with known exact solutions under conditions when the comparatively small $\delta\boldsymbol{\sigma}_n$ term is designated to be zero).

The method for computing $\delta\boldsymbol{\sigma}_n$ in (2) is through an *m* rate computation (higher than the *n* attitude update rate), including down-summing over an *n* cycle to form $\delta\boldsymbol{\sigma}_n$ [13]:

$$\begin{gathered}
\boldsymbol{\phi}_m \approx \Delta\boldsymbol{\alpha}_m + \delta\boldsymbol{\phi}_m \\
\Delta\boldsymbol{\alpha}(t) \equiv \int_{t_{m-1}}^{t} \boldsymbol{\omega}\, d\tau \qquad \Delta\boldsymbol{\alpha}_m = \Delta\boldsymbol{\alpha}(t = t_m) \qquad \delta\boldsymbol{\phi}_m = \frac{1}{2}\int_{t_{m-1}}^{t_m} \Delta\boldsymbol{\alpha}(t) \times \boldsymbol{\omega}\, dt \\
\text{Summing Down :} \\
\boldsymbol{\alpha}_{m-1} = \sum_{m=1}^{M-1} \Delta\boldsymbol{\alpha}_m \qquad \boldsymbol{\sigma}_n = \sum_{m=1}^{M}\left[\frac{1}{2}\left(\boldsymbol{\alpha}_{m-1} \times \Delta\boldsymbol{\alpha}_m\right) + \boldsymbol{\phi}_m\right]
\end{gathered} \tag{4}$$

In (4), rotation vector $\boldsymbol{\phi}_m$, the equivalent of $\boldsymbol{\sigma}_n$ in (1), is computed at the *m* cycle rate, $\Delta\boldsymbol{\alpha}_m$ is the integrated angular rate over an *m* cycle, and *M* is the number of *m* cycles in an *n* cycle. The commonly used equivalent of (4) for real-time digital implementation in a strapdown INS computer is

$$
\begin{gathered}
\phi_{Algo_m} \approx \Delta\alpha_{Algo_m} + \delta\phi_{Algo_m} \\
\Delta\alpha_{Algo_m} = \sum_{j=1}^{L} \Delta\alpha_j \qquad \delta\phi_{Algo_m} = \sum_{j=1}^{L}\sum_{k=1}^{L} C_{j,k}\, \Delta\alpha_j \times \Delta\alpha_k \\
\text{Summing Down:} \\
\alpha_{Algo_{m-1}} = \sum_{m=1}^{M-1} \Delta\alpha_{Algo_m} \qquad \sigma_n = \sum_{m=1}^{M}\left[\frac{1}{2}\left(\alpha_{Algo_{m-1}} \times \Delta\alpha_{Algo_m}\right) + \phi_{Algo_m}\right]
\end{gathered} \tag{5}
$$

where ϕ_{Algo_m}, $\Delta\alpha_{Algo_m}$, $\delta\phi_{Algo_m}$, $\alpha_{Algo_{m-1}}$ are digital algorithmic equivalents of ϕ_m, $\Delta\alpha_m$, $\delta\phi_m$, α_{m-1} in (4), $\Delta\alpha_j$, $\Delta\alpha_k$ are integrated angular rate increment inputs from gyros taken at a higher speed *l* cycle rate (e.g., 2 kHz), *L* is the number of *l* cycles in an *m* cycle, and the $C_{j,k}$ s are constant coefficients designed to meet specified design criteria. A 4-sample algorithm has $L = 4$ for which there are 6 $C_{j,k}$ coefficients (for when $j \neq k$. When $j = k$, $\Delta\alpha_j \times \Delta\alpha_k$ in (5) is zero, and there is no $C_{j,k}$ contribution to $\delta\phi_{Algo_m}$). The general $\delta\phi_{Algo_m}$ form in (5) is derived from a Taylor time-series expansion for angular rate ω [10, Appendix A].

The $\delta\phi_{Algo_m}$ algorithm and associated *l* cycle gyro sampling rate are designed to provide an accurate representation of $\delta\phi_m$ under both sustained angular vibration and transient maneuvers. When multi-axis angular vibrations generate $\Delta\alpha_j$, $\Delta\alpha_k$ increments that are in phase (e.g., sine wave oscillations of the same frequency around two orthogonal gyro input axes), the $\Delta\alpha_j \times \Delta\alpha_k$ product is zero corresponding to a condition when the sensor assembly is rocking and/or spinning about a common axis. When the $\Delta\alpha_j$, $\Delta\alpha_k$ increments are out of phase (e.g., a sine wave oscillation around one gyro axis coupled with a cosine wave oscillation of the same frequency around another gyro axis), the condition is denoted as "coning", and the combined effect in (5) is to generate an average constant $\delta\phi_{Algo_m}$ component that builds with time.

DESIGNING THE $C_{j,k}$ COEFFICIENTS

Four methods have been used to design the $C_{j,k}$ coefficients for the $\delta\phi_{Algo_m}$ calculation in (5): 1) <u>Taylor Time-Series Expansion</u> - Using the original $\delta\phi_{Algo_m}$ formula derivation source [10, Appendix A] based on a truncated Taylor time-series expansion of ω angular rate, 2) <u>Taylor Frequency-Series Optimization</u> - Minimizing the steady $\delta\phi_{Algo_m}$ error under sustained coning vibration by nullifying a truncated Taylor series expansion for $\delta\phi_m - \delta\phi_{Algo_m}$ (the error in $\delta\phi_{Algo_m}$) in powers of vibration frequency [5] – [7], [10, Appendix B], 3) <u>Least-Squares Optimization</u> - Minimizing the integrated weighted squared magnitude of $\delta\phi_m - \delta\phi_{Algo_m}$ under sustained coning vibration over a specified vibration frequency range [10], and 4) <u>Uncompressing Frequency-Based Algorithm Coefficients For Maneuver Accuracy</u> – The Song approach [11] of adapting unused coefficients in a vibration-frequency optimized algorithm for improved accuracy under maneuvers. In general, the 2) or 3) frequency based approaches have more than 100 times smaller error than the 1) Taylor time-series approach under vibration [10, Figs. 4 & 5], whereas under maneuvering, the 2) or 3) frequency based approaches have more than 100 times larger error than the 1) Taylor time-series approach [10, Table 2]. The 4) Song approach retains the accuracy of 2) or 3) under vibration [11, Fig. 2], while improving their maneuvering accuracy to match the 1) capability [11, Table 4]. In this writer's opinion, the [11] Song approach is a uniquely significant contribution to strapdown attitude algorithm design.

EVALUATING ATITUDE ALGORITHM ERROR

Because (1) is exact, all attitude representations selected for (1) will have identical error because each will be an exact translation of the same $\boldsymbol{\sigma}_n$ rotation vector input and its associated errors. Thus, the resulting error in (1) will be a translation of the error in the computation of rotation vector $\boldsymbol{\sigma}_n$. The $\boldsymbol{\sigma}_n$ error arises from two sources:

1) The difference in the approximate (2) integral formula for $\boldsymbol{\sigma}_n$, i.e.,

$\boldsymbol{\sigma}_n \approx \int_{t_{n-1}}^{t_n} \left[\boldsymbol{\omega} + \frac{1}{2} \left(\int_{t_{n-1}}^{t} \boldsymbol{\omega}\, d\tau \right) \times \boldsymbol{\omega} \right] dt$, compared with the integral of the (3) exact

Laning equation: $\boldsymbol{\sigma}_n = \int_{t_{n-1}}^{t_n} \left\{ \boldsymbol{\omega} + \frac{1}{2} \boldsymbol{\sigma} \times \boldsymbol{\omega} + \frac{1}{\sigma^2} \left[1 - \frac{\sigma \sin \sigma}{2 (1 - \cos \sigma)} \right] \boldsymbol{\sigma} \times (\boldsymbol{\sigma} \times \boldsymbol{\omega}) \right\} dt$

2) The error in the digital algorithmic summation for $\delta\boldsymbol{\phi}_{Algo_m}$ in (5) compared with the continuous integral for $\delta\boldsymbol{\phi}_m$ in (4), i.e., $\delta\boldsymbol{\phi}_{Algo_m} - \delta\boldsymbol{\phi}_m$, thereby measuring the effectiveness of the $C_{j,k}$ coefficients in (5).

Note : There is no computational error in the (5) $\Delta\boldsymbol{\alpha}_{Algo_m}$ summation because, in the absence of gyro error, the $\Delta\boldsymbol{\alpha}_j$ inputs from the gyros represent exact integrated-angular-rate-increments.

Error source 1) dominates under large-angle short-time-duration maneuvers, creating an attitude error (measured in μrad per maneuver) that sums with others previously generated throughout a given trajectory. Error source 2) dominates under sustained small amplitude high-frequency angular coning vibration, generating a sustained average attitude error rate (measured in deg/hr), similar to the effect of gyro bias error. Both effects are evaluated as part of attitude algorithm performance analysis. The numerical attitude algorithm error data provided in [10] and [11] (as discussed in the EXISTING ALGORITHM CAPABILITIES section to follow) is based on the previous error source definitions.

NEW ATTITUDE ALGORITHM ACCURACY REQUIREMENTS

There is a common misconception that there will be a continuing need for ever-more-accurate strapdown attitude algorithms for compatibility with emerging future precision gyro capabilities. The fallacy in this argument is that it doesn't account for the fundamental INS free-inertial attitude accuracy limit imposed by INS gravity model uncertainty: 20 μg contributing 20 μrad (0.0011 deg) to attitude error (through attitude/velocity/position gravity-model computational coupling). For a traditional 1 nmph (nautical mile per hour) INS, gravity model uncertainty has only a minor impact in the overall system error budget. For a higher accuracy future strapdown INS (e.g., 0.1 nmph), however, gravity model error becomes a significant error budget contributor. Thus, any forecasted development of INS technology having less than 0.1 nmph error is unlikely. It follows that the need for new gyros with higher accuracy than required for 0.1 nmph navigation is also unlikely, as is a corresponding requirement for new higher accuracy strapdown attitude algorithms.

In a standard strapdown INS, the often referenced 1 nmph position error growth is produced by an average attitude computational bias of 0.017 deg/hr (of which 0.01 deg/hr is from gyro bias). This is accompanied by a low frequency bounded "Schuler" oscillation [12] in attitude (i.e., one cycle per 84 minute period) on the order of 0.004 deg amplitude [9, Eq. (13.3.2-25) for horizontal attitude error γ_H as function of horizontal gyro bias $\delta\omega_{IB_H}$ and Schuler frequency ω_S, both in rad/sec], generated by 0.01 deg/hr gyro bias, gyro scale-factor error (order of 5 ppm – parts per million), accelerometer bias (order of 40 μg), sensor-to-sensor misalignment (order of 10 μrad), and a negligible portion for computational algorithm error. Budgeting 1% of the 0.004 deg for computational process error, the attitude algorithm error allowance becomes 0.00017 deg/hr and 0.00004 deg. For a future 0.1 nmph strapdown INS, the algorithm error allowance would be an order of magnitude smaller or 0.000017 deg/hr and 0.000004 deg (including 1 μrad allowance for sensor-to-sensor misalignment – a physically minimum achievable limit for future sensor-assembly technology). The 0.000017 deg/hr requirement would be targeted against algorithm performance under sustained severe angular vibration; the 0.000004 deg requirement against performance under a composite of short term (e.g., 2 sec) extreme angular maneuvers over a given trajectory.

EXISTING ALGORITHM CAPABILITIES

The computational structure of many current-day strapdown attitude algorithms is represented by (1) - (5) shown previously. Attitude algorithms defined in [4] – [11] and [13] fit this structure, being distinguishable from each other by the particular $C_{j,k}$ coefficient numerical values used in the (5) coning correction algorithm.

To illustrate the accuracy of a traditional algorithm under vibration, consider a worst-case 7.6 grms linear vibration into a typical isolator-mounted sensor assembly having 50 Hz undamped natural linear isolator resonance frequency and 0.125 damping ratio. For worst-case 2% isolator mismatch and 0.5 % center-of-mass offset, [13] shows that 10 deg/hr average sensor assembly coning rate would be produced, i.e., the average value of $\delta\sigma_n$ in (2). Fig. 2 of [11] shows that for a 4-sample Song extended least-squares frequency-based algorithm operating at a 1 kHz high-speed computation rate (in (5), the $\Delta\alpha_j$ sample rate), the algorithm error would be 0.000002 deg/hr, easily satisfying the 0.000017 deg/hr requirement discussed previously. The error would be 10 times smaller using a 5-sample algorithm.

As an example of the maneuvering accuracy provided by existing strapdown attitude algorithms, consider the extreme 2 second angular rate maneuver in Fig. 1 of [11]. Table 4 of [11] shows that for a 0.005 sec (i.e., 200 Hz) attitude update *n* cycle rate in (1), the angular error generated during the maneuver by a Song-type algorithm would be on the order of 0.001 μrad (0.6×10^{-7} deg). Ten of such maneuvers over a particular flight would generate a combined error on the order of 0.01 μrad (0.6×10^{-6} deg), negligible compared to the previously defined 0.000004 deg requirement. The error could be reduced further by increasing the attitude update rate, e.g., doubling the attitude update rate would decrease the error over the maneuver by a factor of 16 [8, Eq. (44) for algorithm c].

The previous examples demonstrate that existing attitude updating algorithms have more than enough accuracy for achieving 0.000017 deg/hr average accuracy under vibration and 0.000004 deg accuracy under maneuvers, more than would ever be required in a future strapdown INS. Thus, higher accuracy touted for new attitude algorithms would have virtually no benefit in the overall error budget of a future strapdown INS.

CONCLUSION

Attitude accuracy requirements for forecasted future gyros in unaided inertial navigation applications will be on the order of 0.0002 deg under maneuvering and 0.001 deg/hr under steady flight. Higher accuracy would be of no benefit because it would be overshadowed by the effect of gravity uncertainty. Accuracy requirements in aided applications is even less because the aiding will compensate gyro errors. Thus, future gyro development will most likely focus on reduced cost or other improvements for the same accuracy. For compatibility, attitude computation algorithm error under maneuvering should be one or two orders of magnitude smaller, e.g., 0.00004 to 0.000004 deg under worst case maneuvering and 0.0002 to 0.00002 deg/hr under worst case vibration. Current attitude algorithms easily meet these requirements and can be extended if needed in future special applications by increasing algorithm update rates or the number of gyro samples per update.

On recounting the history of strapdown algorithm development, I am often asked "if that was the way the real algorithm writers did it?" Reflecting on modern-day algorithms compared to those in the early strapdown days, my response typically paraphrases what I learned many years ago from an experienced old-timer:

> "Most of the original algorithm writers had tricks of their own. One for instance used truncated attitude algorithms and depended on normalization/orthogonalization routines to partially compensate the truncation errors. Another used ultra high-speed updating with simpler less sophisticated algorithms. And there were some who liked different algorithms for different applications. But <u>one</u> good two-speed algorithm with exact attitude updating is all you need today if you know how to use it. Good for navigating anything to where you want it to go." [Savage, P.G., "Blazing Gyros – The Movie", SAI-WBN-14016, May 25, 2016 http://www.strapdownassociates.com/Blazing%20Gyros%20-%20The%20Movie.pdf]

REFERENCES (IN ORDER OF PUBLICATION)

[1] Laning, Jr., J.H., "The Vector Analysis of Finite Rotations and Angles", Massachusetts Institute of Technology, Cambridge, Instrumentation Laboratory Special Report 6398-S-3, 1949

[2] Savage, P.G., “A New Second-Order Solution for Strapped-Down Attitude Computation”, AIAA Joint Automatic Control Conference, Seattle, Washington, August 15-17, 1966.

[3] Bortz J. E., "A New Concept In Strapdown Inertial Navigation", MIT Doctoral Thesis, June, 1969.

[4] Jordan, J. W., “An Accurate Strapdown Direction Cosine Algorithm”, NASA TN-D-5384, September 1969.

[5] Miller, R., “A New Strapdown Attitude Algorithm”, *AIAA Journal Of Guidance, Control, And Dynamics*, Vol. 6, No. 4, July-August 1983, pp. 287-291.

[6] Ignagni, M. B., “Optimal Strapdown Attitude Integration Algorithms”, *AIAA Journal Of Guidance, Control, And Dynamics*, Vol. 13, No. 2, March-April 1990, pp. 363-369.

[7] Ignagni, M. B., “Efficient Class Of Optimized Coning Compensation Algorithms”, *AIAA Journal Of Guidance, Control, And Dynamics*, Vol. 19, No. 2, March-April 1996, pp. 424-429.

[8] Savage, P. G., “A Unified Mathematical Framework For Strapdown Attitude Design”, *AIAA Journal Of Guidance, Control, And Dynamics*, Vol. 29, No. 2, Mar - Apr 2006, pp. 237-249.

[9] Savage, Paul G., *Strapdown Analytics - Second Edition*, Chapter 13, Strapdown Associates, Inc., Maple Plain, Minnesota, 2007.

[10] Savage, P.G., “Coning Algorithm Design By Explicit Frequency Shaping”, *AIAA Journal Of Guidance, Control, And Dynamics*, Vol. 33, No. 4, Jul-Aug 2010, pp. 1123-1132.

[11] Song, M., Wu, W., and Pan, X. “Approach To Recovering Maneuver Accuracy In Classical Coning Algorithms”, *AIAA Journal Of Guidance, Control, And Dynamics*, Vol. 36, No. 6, Nov-Dec 2013, pp. 1872-1881.

[12] Savage, P.G. “Schuler Oscillations”, Strapdown Associates, Inc., SAI WBN-14003, Jun 27, 2014, http://strapdownassociates.com/Schuler%20Oscillations.pdf.

[13] Savage, P. G., “Down-Summing Rotation Vectors For Attitude Updating”, SAI-WBN-14019, Jul 16, 2017, Strapdown Associates, Inc., Jul 16, 2017, http://strapdownassociates.com/Rotation%20Vector%20Down_Summing.pdf